

Answer any 2 questions

Q1. (i) Represent the complex number $1 - i\sqrt{3}$ in the polar form $re^{i\theta}$.

Answer:

$$r = \sqrt{1^2 + (-\sqrt{3})^2} = 2$$
$$\theta = \arctan(-\sqrt{3}) = -\frac{\pi}{3} = \frac{5\pi}{3}$$

[4 Marks]

(ii) Express $\cos(1 - i\sqrt{3})$ and $\sin(1 - i\sqrt{3})$ in the form $x + iy$ where x and y are real numbers.

Answer: Using

$$\cos z = \cos x \cosh y - i \sin x \sinh y$$
$$\sin z = \sin x \cosh y + i \cos x \sinh y$$

where $z = x + iy = 1 - i\sqrt{3}$,

$$\cos z = \cos(1) \cosh(-\sqrt{3}) - i \left(\sin(1) \sinh(-\sqrt{3}) \right) = 1.5748 + i2.3037$$
$$\sin z = \sin(1) \cosh(-\sqrt{3}) + i \left(\cos(1) \sinh(-\sqrt{3}) \right) = 2.4525 - i1.4792$$

[5 Marks]

(iii) Find all the values of $\ln(1 - i\sqrt{3})$ and also the principal value $\text{Ln}(1 - i\sqrt{3})$.

Answer: Using

$$\ln z = \ln |z| + i \arg z \pm 2n\pi i$$

where $z = x + iy = 1 - i\sqrt{3}$,

$$\ln(1 - i\sqrt{3}) = \ln 2 - i\frac{\pi}{3} \pm 2n\pi i = 0.69315 + i(\pm 2n\pi - 1.0472)$$
$$\text{Ln}(1 - i\sqrt{3}) = \ln 2 - i\frac{\pi}{3}$$

[7 Marks]

(iv) Find all complex numbers z such that $\sin z = i/2$.

Answer: Using $\sin z = \frac{1}{2i}(e^{iz} - e^{-iz})$,

$$\begin{aligned}\sin z &= \frac{i}{2} = -\frac{1}{2i} \\ \implies -1 &= e^{iz} - e^{-iz} \\ -e^{iz} &= e^{2iz} - 1 \\ (e^{iz})^2 + e^{iz} - 1 &= 0\end{aligned}$$

$$\begin{aligned}e^{iz} &= \frac{-1 \pm \sqrt{1^2 - 4(1)(-1)}}{2(1)} \\ &= -\frac{1}{2} \pm \frac{\sqrt{5}}{2} \\ iz &= \ln \frac{-1 \pm \sqrt{5}}{2} + 2n\pi i\end{aligned}$$

Using

$$\ln z = \ln |z| + i \arg z + 2n\pi i$$

$$iz_1 = \ln \frac{-1 + \sqrt{5}}{2}$$

$$iz_1 = \ln(|0.61803|) + 0i \pm 2n\pi i$$

$$iz_1 = \ln(|0.61803|) \pm 2n\pi i$$

$$iz_1 = -0.48121 \pm 2n\pi i$$

$$z_1 = -0.48121/i \pm 2n\pi$$

$$z_1 = \pm 2n\pi + 0.48121i$$

or

$$iz_2 = \ln \frac{-1 - \sqrt{5}}{2}$$

$$iz_2 = \ln(|-1.6180|) + \pi i \pm 2n\pi i$$

$$iz_2 = \ln(|-1.6180|) + \pi i \pm 2n\pi i$$

$$iz_2 = 0.48121 + \pi i \pm 2n\pi i$$

$$z_2 = 0.48121/i + \pi \pm 2n\pi$$

$$z_2 = (\pm 2n + 1)\pi - 0.48121i$$

[9 Marks]

Q2. (i) Use the Cauchy-Riemann equations to show that the function

$$f(x + iy) = (x^2y + 1) + i(xy^2 - 1) \quad x, y \in \mathbb{R}$$

is continuously differentiable along the lines $y = \pm ix$ and find the derivative $f'(z)$ at the point $z = 1 - i$.

Answer: Cauchy-Reimann:

$$\frac{\partial u}{\partial x} = +\frac{\partial v}{\partial y} \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

$$u(x, y) = \Re(f(x, y)) = x^2y + 1$$

$$v(x, y) = \Im(f(x, y)) = xy^2 - 1$$

$$\begin{aligned} \frac{\partial u}{\partial x} &= 2xy & \frac{\partial v}{\partial y} &= 2xy & (2xy &= 2xy \forall x, y) \\ \frac{\partial u}{\partial y} &= x^2 & \frac{\partial v}{\partial x} &= y^2 & \implies x^2 &= -y^2 \\ & & & & \implies x^2 &= i^2y^2 \end{aligned}$$

$$f'(1 - i) = \left. \frac{\partial u}{\partial x} \right|_{1-i} + i \left. \frac{\partial v}{\partial x} \right|_{1-i} = 2(1)(-1) + i(-1)^2 = -2 + i$$

[9 Marks]

(ii) Show that

$$u(x, y) = x^2 - y^2 \quad x, y \in \mathbb{R}$$

is a harmonic function and find the analytic function $f(z)$ for which $u(x, y)$ is the real part, where $z = x + iy$. Write $f(z)$ as a function of z .

Answer: Laplace: $\nabla_2 u = 0$

$$\begin{aligned} \frac{\partial}{\partial x} \frac{\partial u}{\partial x} &= \frac{\partial}{\partial x} (+2x) = +2 \\ \frac{\partial}{\partial y} \frac{\partial u}{\partial y} &= \frac{\partial}{\partial y} (-2y) = -2 \\ \implies \nabla_2 u &= 0 \end{aligned}$$

Cauchy-Reimann:

$$\frac{\partial u}{\partial x} = +\frac{\partial v}{\partial y} \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

$$\begin{aligned} \frac{\partial u}{\partial x} &= \frac{\partial v}{\partial y} = 2x \\ \implies v &= \int 2x \partial y = 2xy + h(x) \\ \implies \frac{\partial v}{\partial x} &= 2y + \frac{dh}{dx} \\ \text{From Cauchy-Reimann} \quad -\frac{\partial u}{\partial y} &= \frac{\partial v}{\partial x} = 2y \\ \implies \frac{dh}{dx} &= 0 \implies h = C \end{aligned}$$

$$\begin{aligned} f(x + iy) &= u(x, y) + iv(x, y) \\ &= (x^2 - y^2) + i(2xy + C) \\ &= (x + iy)^2 + iC \\ f(z) &= z^2 + iC \end{aligned}$$

[10 Marks]

(iii) Calculate the line integral

$$\int_C (z^2 + 1) dz$$

where C is the straight line $C : z(t) = t(1 + i) : 0 \leq t \leq 1$

Answer:

$$\begin{aligned} \int_c f(z) dz &= \int_0^1 f(z(t)) \dot{z}(t) dt \\ \dot{z}(t) &= 1 + i \end{aligned}$$

$$\begin{aligned} f(z(t)) &= (z^2(t) + 1) \\ &= (t(1 + i))^2 + 1 \\ &= i2t^2 + 1 \end{aligned}$$

$$\begin{aligned} \int_0^1 f(z(t)) dz &= \int_0^1 (i2t^2 + 1)(1 + i) dt \\ &= \int_0^1 (2t^2 i + 2t^2 i^2 + 1 + i) dt \\ &= \int_0^1 (2t^2(i - 1) + 1 + i) dt \\ &= \left[2(i - 1) \frac{t^3}{3} + (i + 1)t \right]_0^1 \\ &= \frac{2}{3}(i - 1) + (i + 1) \\ &= \frac{5}{3}i + \frac{1}{3} \end{aligned}$$

[6 Marks]

Q3. (i) Evaluate the following integral

$$\int_C \frac{e^{iz}}{(z - \pi)^3} dz$$

where C is the circle with radius 4 centred on the origin (traversed once in the counter-clockwise direction).

[13 Marks]

(ii) Show that

$$\int_0^{2\pi} \frac{1}{\sqrt{17 + \cos \theta}} d\theta = \frac{\pi}{2}$$

Answer: No one who sat the exam attempted this question.

[12 Marks]

END