Maths 2B solutions for tutorial questions on second order linear differential equations, solved in the time domain.

May 5, 2008

Please report any errors to g.bevan@mech.gla.ac.uk

Questions

For each of the systems below, find

- the complementary function
- the particular integral
- the full general solution

and sketch the response. Also find the damping ratio and, if the system is under-damped ($\zeta < 1$), the frequency and amplitude of oscillation.

(1)	$\ddot{x} + \dot{x} + x = 0$	$x(0) = 1$	$\dot{x}(0) = 0$
(2)	$\ddot{y} + 2\dot{y} + y = 0$	$y(0) = 1$	$\dot{y}(0) = 0$
(3)	$\ddot{z} + 3\dot{z} + z = 0$	$z(0) = 1$	$\dot{z}(0) = 0$
(4)	$\dot{p} + p = 0$	$p(0) = 1$	$\dot{p}(0) = 0$
(5)	$\ddot{q} + q = 0$	$q(0) = 1$	$\dot{q}(0) = 0$
(6)	$\ddot{\theta} + 2\dot{\theta} + \theta = t$	$\theta(0) = 0$	$\dot{\theta}(0) = 0$
(7)	$\ddot{\phi} + 3\dot{\phi} + \phi = e^{-3t}$	$\phi(0) = 2$	$\dot{\phi}(0) = 0$
(8)	$\ddot{\psi} + 4\dot{\psi} + \psi = \cos 2t$	$\psi(0) = 0$	$\dot{\psi}(0) = 0$

Question 1

(Q.1) $$\ddot{x} + \dot{x} + x = 0$$

(IC.1a) $$x(0) = 1$$
(IC.1b) $$\dot{x}(0) = 0$$

Particular Integral

(PI.1)
$$\begin{aligned} x_{PI}(t) &= 0 \\ \implies \dot{x}_{PI}(t) &= 0 \\ \implies \ddot{x}_{PI}(t) &= 0 \end{aligned}$$

Equation (Q.1) is homogeneous so there is no particular integral.

Complementary Function

(Aux.1) $$\lambda^2 + \lambda + 1 = 0$$

$$\begin{aligned} \lambda &= \frac{-1 \pm \sqrt{1^2 - 4 \times 1 \times 1}}{2 \times 1} \\ &= -\frac{1}{2} \pm j\frac{\sqrt{3}}{2} \end{aligned}$$ (Roots.1)

A complex pair of roots ($\lambda = \alpha \pm j\beta$), so the complementary function is of the form

$$\begin{aligned} x_{CF}(t) &= e^{\alpha t}\left(A\cos\beta t + B\sin\beta t\right) \\ &= e^{-\frac{1}{2}t}\left(A\cos\frac{\sqrt{3}}{2}t + B\sin\frac{\sqrt{3}}{2}t\right) \end{aligned}$$ (CF.1)

General solution

The general solution is given by the sum of the complementary function and the particular integral.

$$\begin{aligned} x(t) &= x_{CF}(t) + x_{PI}(t) \\ &= e^{-\frac{1}{2}t}\left(A\cos\frac{\sqrt{3}}{2}t + B\sin\frac{\sqrt{3}}{2}t\right) \end{aligned}$$ (GS.1)

The constants A and B are found by applying the initial conditions. First differentiate (GS.1) using the product rule to get $\dot{x}(t)$

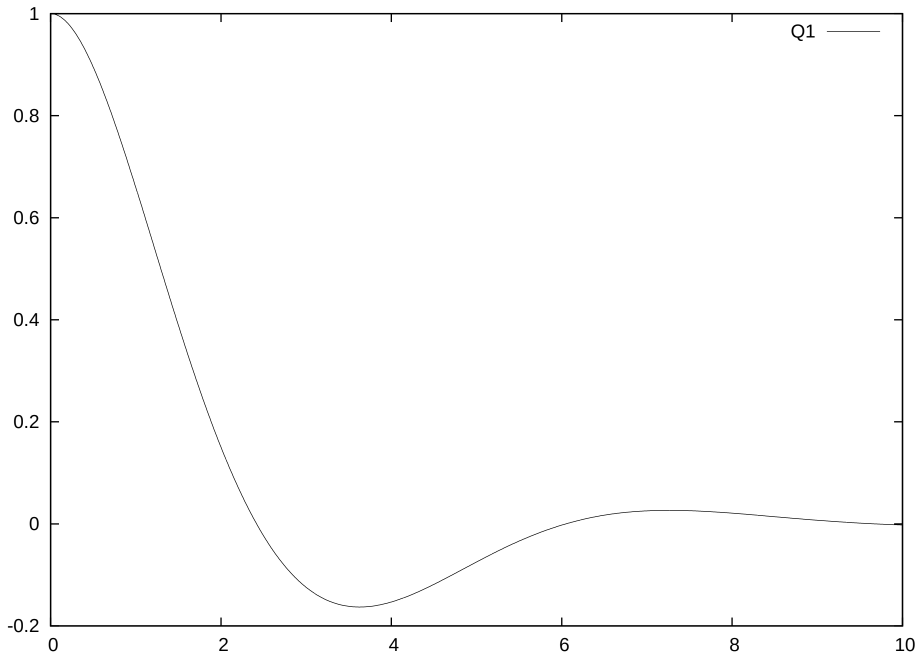
(GS'.1)
$$\begin{aligned} \dot{x}(t) = -\frac{1}{2}e^{-\frac{1}{2}t}&\left(A\cos\frac{\sqrt{3}}{2}t + B\sin\frac{\sqrt{3}}{2}\right) \\ &+ e^{-\frac{1}{2}t}\left(-A\frac{\sqrt{3}}{2}\sin\frac{\sqrt{3}}{2}t + B\frac{\sqrt{3}}{2}\cos\frac{\sqrt{3}}{2}t\right) \end{aligned}$$

Then set $t = 0$ in (GS.1) and (GS'.1) and equate to (IC.1a) and (IC.1b) respectively

$$\begin{aligned} x(0) &= A && = 1 \\ \dot{x}(0) &= -\frac{1}{2}A + \frac{\sqrt{3}}{2}B && = 0 \end{aligned}$$

Solving the simultaneous equations gives $A = 1$ and $B = \frac{1}{\sqrt{3}}$.

(FGS.1) $$x(t) = e^{-\frac{1}{2}t}\left(\cos\frac{\sqrt{3}}{2}t + \frac{1}{\sqrt{3}}\sin\frac{\sqrt{3}}{2}t\right)$$



Damping coefficient and frequency

The Auxiliary Equation (Aux.1) had a complex pair of roots, so the system is under-damped and will oscillate. The coefficients can be equated to the damping ratio ζ and frequency ω in the standard form

$$\ddot{x}^2 + 2\zeta\omega\dot{x} + \omega^2 x = 0$$

Hence $2\zeta\omega = 1$ and $\omega^2 = 1$, so the frequency of oscillation is $\omega = 1[rad/s]$ and the damping ratio is $\zeta = \frac{1}{2}$.

Amplitude of oscillation

The amplitude and phase lag are found by converting the General Solution into the form

$$x(t) = Me^{\alpha t}\cos(\beta t + \phi)$$

$$M = \sqrt{A^2 + B^2} = \frac{2}{\sqrt{3}}$$
$$\phi = \arctan\frac{A}{B} = \frac{\pi}{3}[rad]$$

So the amplitude as a function of time is

(Mag.1) $$Me^{\alpha t} = \frac{2}{\sqrt{3}}e^{-\frac{1}{2}t}$$

1 Question 2

(Q.2) $$\ddot{y} + 2\dot{y} + y = 0$$

(IC.2a) $$y(0) = 1$$
(IC.2b) $$\dot{y}(0) = 0$$

1.1 Particular Integral

As in Q.1, the equation is homogeneous so the Particular Integral is zero

(PI.2) $$y_{PI}(t) = \dot{y}_{PI}(t) = \ddot{y}_{PI}(t) = 0$$

1.2 Complementary Function

(Aux.2) $$\lambda^2 + 2\lambda + 1 = 0$$

(Roots.2) $$\lambda = -1$$

A repeated real root, so the complementary function is of the form

(CF.2) $$\begin{aligned} y_{CF}(t) &= (A + Bt)e^{\lambda t} \\ &= (A + Bt)e^{-t} \end{aligned}$$

General Solution

The general solution $y(t)$ is the sum of $y_{CF}(t)$ and $y_{PI}(t)$

(GS.2) $$y(t) = (A + Bt)e^{-t}$$

The constants A and B are found by applying the initial conditions. First differentiate (GS.2) using the product rule to get $\dot{y}(t)$

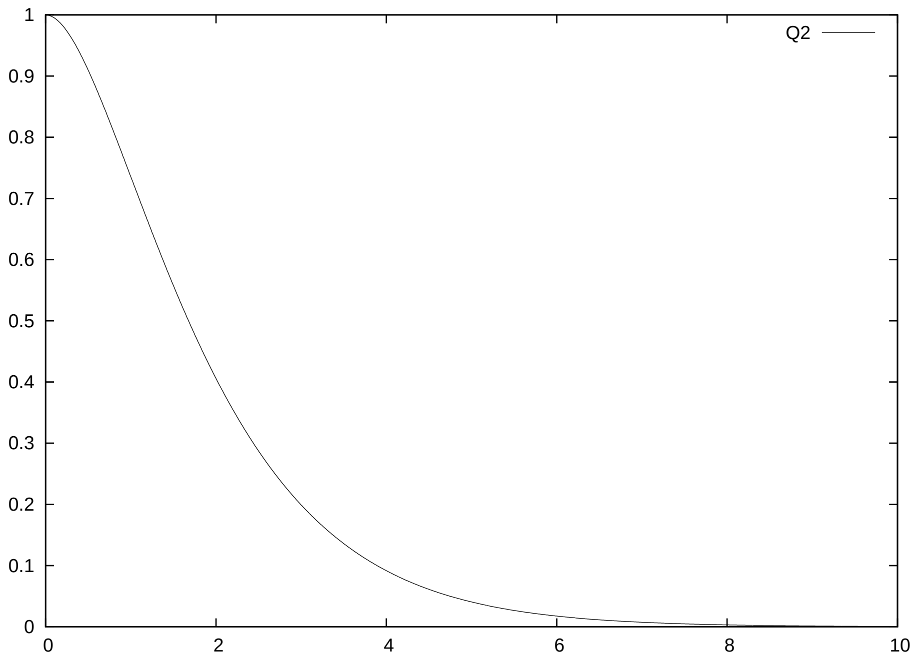
(GS'.2) $$\dot{y}(t) = (B - A - Bt)e^{-t}$$

Then set $t = 0$ in (GS.2) and (GS'.2) and equate to (IC.2a) and (IC.2b) respectively

$$\begin{aligned} y(0) = A &= 1 \\ \dot{y}(0) = B - A &= 0 \end{aligned}$$

Solving the simultaneous equations gives $A = B = 1$

(FGS.2) $$y(t) = (1 + t)e^{-t}$$



Damping ratio

The Auxiliary Equation (Aux.2) had one repeated real root so the system is critically damped ($\zeta = 1$) and will not oscillate.

Question 3

(Q.3) $$\ddot{z} + 3\dot{z} + z = 0$$

(IC.3a) $$z(0) = 1$$
(IC.3b) $$\dot{z}(0) = 0$$

Particular Integral

As in Q.1 and Q.2, the equation is homogeneous so the Particular Integral is zero

(PI.3) $$z_{PI} = \dot{z}_{PI} = \ddot{z}_{PI} = 0$$

Complementary Function

(Aux.3) $$\lambda^2 + 3\lambda + 1 = 0$$

(Roots.3) $$\lambda = \frac{-3 \pm \sqrt{5}}{2}$$

Two real roots, so the complementary function is of the form

$$\begin{aligned} z_{CF}(t) &= Ae^{\lambda_1 t} + Be^{\lambda_2 t} \\ &= Ae^{\frac{-3+\sqrt{5}}{2}t} + Be^{\frac{-3-\sqrt{5}}{2}t} \end{aligned}$$ (CF.3)

General Solution

The general solution $z(t)$ is the sum of the Complementary Function and Particular Integral

$$\begin{aligned} z(t) &= z_{CF}(t) + z_{PI}(t) \\ &= Ae^{\frac{-3+\sqrt{5}}{2}t} + Be^{\frac{-3-\sqrt{5}}{2}t} \end{aligned}$$ (GS.3)

The constants A and B are found by applying the initial conditions. First differentiate (GS.3) to get $\dot{z}(t)$

(GS'.3) $$\dot{z}(t) = \frac{-3+\sqrt{5}}{2}Ae^{\frac{-3+\sqrt{5}}{2}t} + \frac{-3-\sqrt{5}}{2}Be^{\frac{-3-\sqrt{5}}{2}t}$$

Then set $t = 0$ and equate (GS.3) and (GS'.3) to the initial conditions (IC.3a) and (IC.3b) respectively

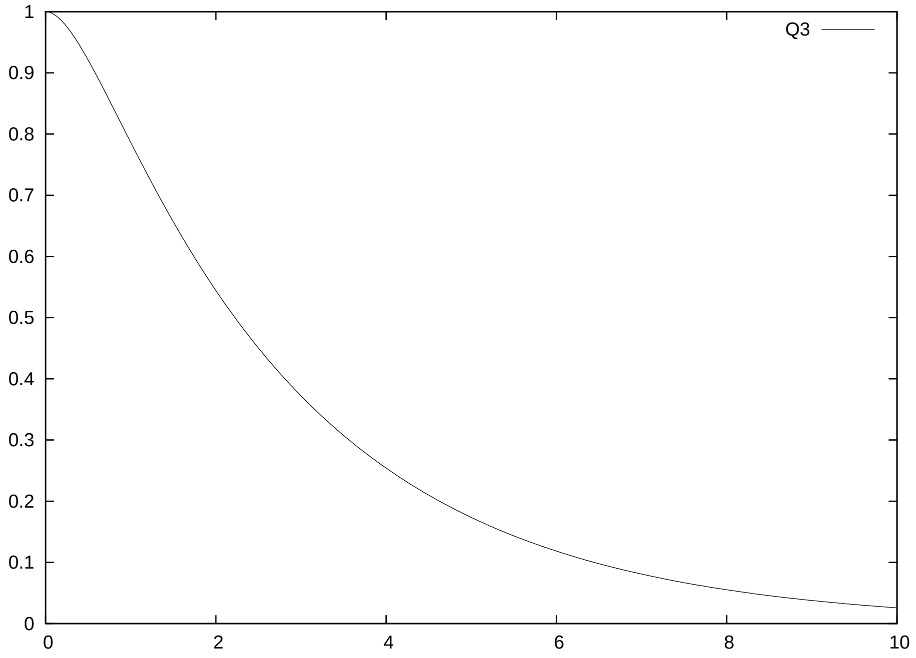
$$\begin{aligned} z(0) &= A + B & &= 1 \\ \dot{z}(0) &= \frac{-3+\sqrt{5}}{2}A + \frac{-3-\sqrt{5}}{2}B & &= 0 \end{aligned}$$

Solving the simultaneous equations gives $A = \frac{1}{2} + \frac{3}{2\sqrt{5}}$ and $B = \frac{1}{2} - \frac{3}{2\sqrt{5}}$

(FGS.3) $$z(t) = \left(\frac{1}{2} + \frac{3}{2\sqrt{5}}\right)e^{\frac{-3+\sqrt{5}}{2}t} + \left(\frac{1}{2} - \frac{3}{2\sqrt{5}}\right)e^{\frac{-3-\sqrt{5}}{2}t}$$

Damping ratio

The Auxiliary Equation (Aux.3) has two real roots so the system is over damped ($\zeta > 1$) and will not oscillate.

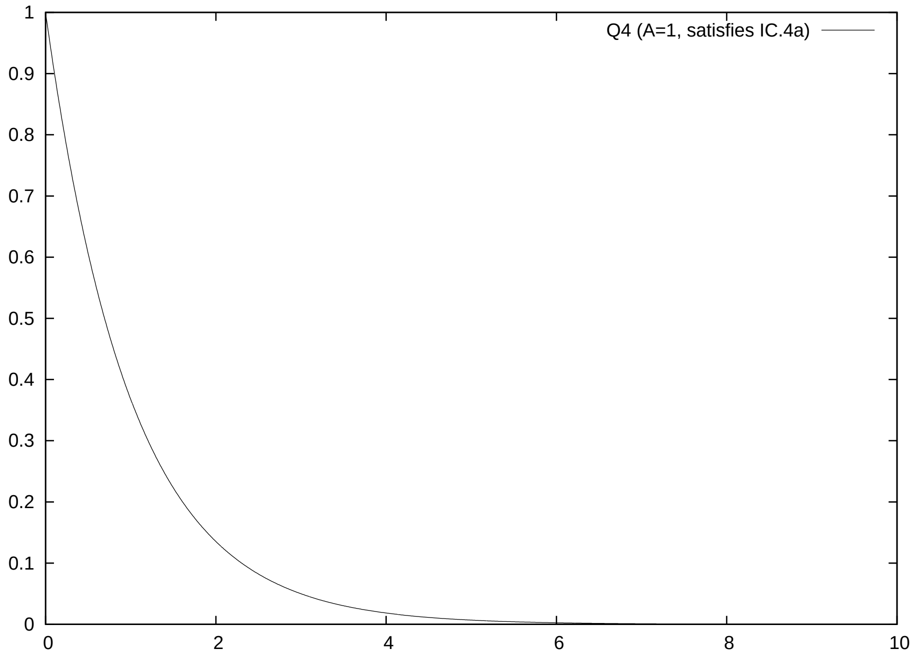


Question 4

(Q.4) $$\dot{p} + p = 0$$

(IC.4a) $$p(0) = 1$$
(IC.4b) $$\dot{p}(0) = 0$$

Equation (Q.4) is a *first order* equation. The solution is simply $p(t) = Ae^{-t}$. But note that the initial conditions are inconsistent. There is no value of A that satisfies both (IC.4a) and (IC.4b) simultaneously. A first order system has only one degree of freedom so cannot satisfy two arbitrary initial conditions.



Question 5

(Q.5) $$\ddot{q} + q = 0$$

(IC.5a) $$q(0) = 1$$
(IC.5b) $$\dot{q}(0) = 0$$

Particular Integral

The equation is homogeneous so the Particular Integral is zero

(PI.5) $$q_{PI}(t) = \dot{q}_{PI}(t) = \ddot{q}_{PI}(t) = 0$$

Complementary Function

(Aux.5) $$\lambda^2 + 1 = 0$$

(Roots.5) $$\lambda = \pm j$$

A complex pair of roots (with zero real parts) so the the complementary function is of the form

(CF.5) $$\begin{aligned} q_{CF}(t) &= e^{\alpha t}(A\cos\beta t + B\sin\beta t) \\ &= A\cos t + B\sin t \end{aligned}$$

General Solution

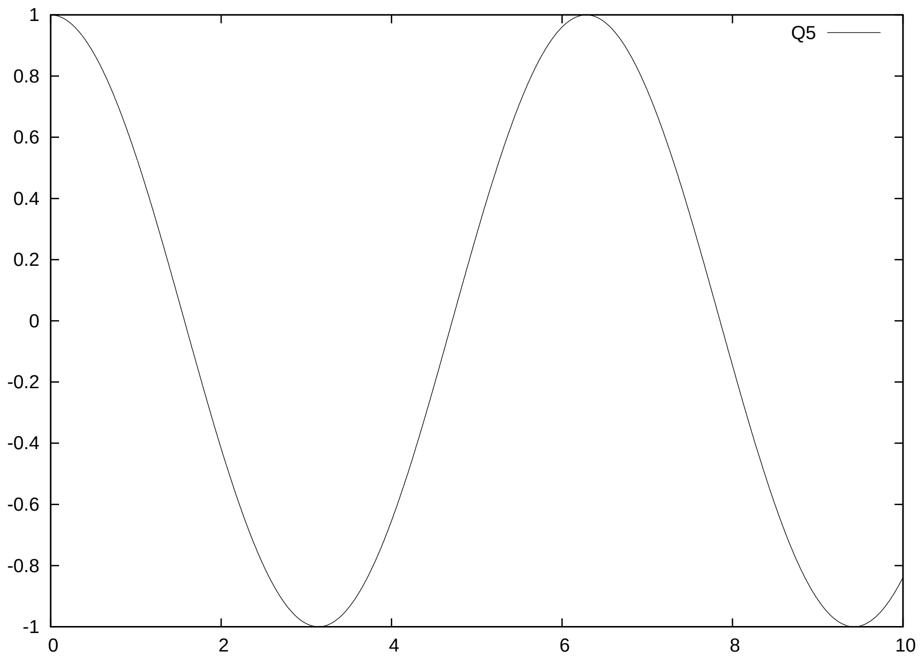
(GS.5) $$\begin{aligned} q(t) &= q_{CF}(t) + q_{PI}(t) \\ &= A\cos t + B\sin t \end{aligned}$$

Find A and B by differentiating (GS.5) to get $\dot{q}(t)$ and applying initial conditions (IC.5a) and (IC.5b)

(GS'.5) $$\dot{q}(t) = -A\sin t + B\cos t$$

$$\begin{aligned} q(0) = A &= 1 \\ \dot{q}(0) = B &= 0 \end{aligned}$$

(FGS.5) $$q(t) = \cos t$$



Damping Ratio and amplitude of oscillation

There is no damping term in (Q.5) which leads to the complex pair of roots with no real part. The system is undamped ($\zeta = 0$) and will oscillate indefinitely with unity amplitude.

Question 6

(Q.6) $$\ddot{\theta} + 2\dot{\theta} + \theta = t$$

(IC6.a) $$\theta(0) = 0$$

(IC.6b) $$\dot{\theta}(0) = 0$$

Particular Integral

$$\begin{aligned} \theta_{PI}(t) &= Ct + D \\ \implies \dot{\theta}_{PI}(t) &= C \\ \implies \ddot{\theta}_{PI}(t) &= 0 \end{aligned}$$

The coefficients C and D are found by substituting the Particular Integral and its derivatives into the original equation (Q.6) and equating coefficients

$$1 \times (0) + 2 \times (C) + 1 \times (Ct + D) = t$$

$$\begin{aligned} t^0 &: 2C + D = 0 \\ t^1 &: C = 1 \end{aligned}$$

So $C = 1$ and $D = -2$, giving the particular integral

(PI.6) $$\theta_{PI}(t) = t - 2$$

Complementary Function

(Aux.6) $$\lambda^2 + 2\lambda + 1 = 0$$

(Roots.6) $$\lambda = -1$$

A repeated real root, so the complementary function is of the form

(CF.6) $$\theta_{CF}(t) = (A + Bt)e^{-t}$$

General Solution

$$\begin{aligned} \theta(t) &= \theta_{CF}(t) + \theta_{PI}(t) \\ &= (A + Bt)e^{-t} + (t - 2) \end{aligned}$$ (GS.6)

Find A and B by applying initial conditions

(GS'.6) $$\dot{\theta}(t) = (B - A - Bt)e^{-t} + 1$$

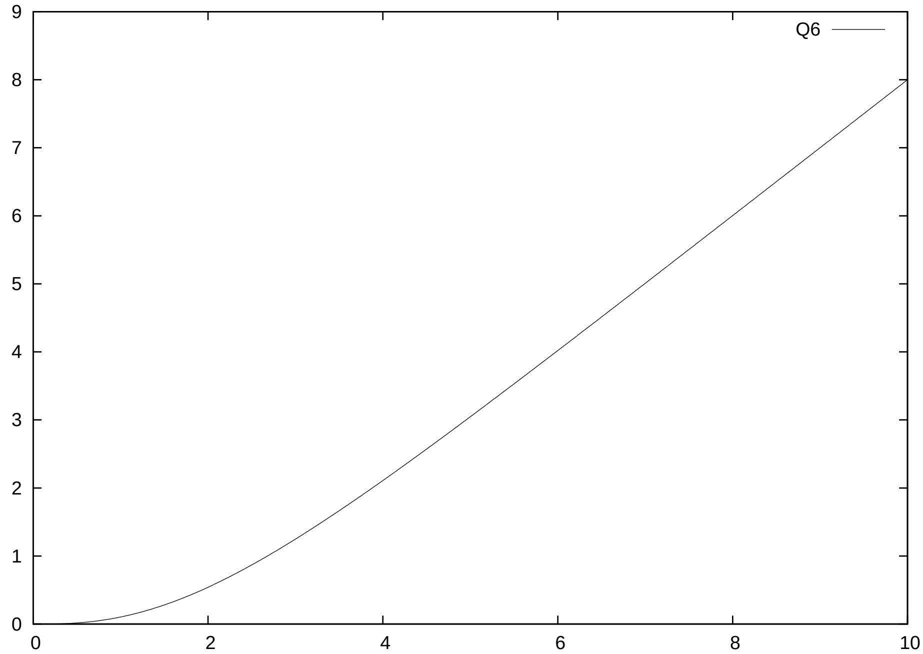
$$\begin{aligned} \theta(0) = A - 2 &= 0 \\ \dot{\theta}(0) = B - A + 1 &= 0 \end{aligned}$$

So $A = 2$ and $B = 1$

(FGS.6) $$\theta(t) = (2 + t)e^{-t} + (t - 2)$$

Damping ratio and amplitude of oscillation

The Auxiliary Equation (Aux.6) has two real roots so the system is over damped ($\zeta > 1$) and will not oscillate.



Question 7

(Q.7) $$\ddot{\phi} + 3\dot{\phi} + \phi = e^{-3t}$$

(IC.7a) $$\phi(0) = 2$$

(IC.7b) $$\dot{\phi}(0) = 0$$

Particular Integral

$$\begin{aligned} \phi_{PI} &= Ce^{-3t} \\ \implies \dot{\phi}_{PI} &= -3Ce^{-3t} \\ \implies \ddot{\phi}_{PI} &= 9Ce^{-3t} \end{aligned}$$

Find C by substituting each term into equation (Q.7) and equating coefficients

(9) $$1 \times (9Ce^{-3t}) + 3 \times (-3Ce^{-3t}) + 1 \times (Ce^{-3t}) = e^{-3t}$$

So $C = 1$

(PI.7) $$\phi_{PI}(t) = e^{-3t}$$

Complementary Function

(Aux.7) $$\lambda^2 + 3\lambda + 1 = 0$$

(Roots.7) $$\lambda = \frac{-3 \pm \sqrt{5}}{2}$$

Two real roots, so the complementary function is of the form

(CF.7) $$\phi_{CF}(t) = Ae^{\frac{-3+\sqrt{5}}{2}t} + Be^{\frac{-3-\sqrt{5}}{2}t}$$

1.3 General Solution

$$\phi(t) = \phi_{CF}(t) + \phi_{PI}(t)$$

(GS.7) $$= Ae^{\frac{-3+\sqrt{5}}{2}t} + Be^{\frac{-3-\sqrt{5}}{2}t} + e^{-3t}$$

Find A and B by applying the initial conditions

(GS'.7) $$\dot{\phi}(t) = \frac{-3+\sqrt{5}}{2}Ae^{\frac{-3+\sqrt{5}}{2}t} + \frac{-3-\sqrt{5}}{2}Be^{\frac{-3-\sqrt{5}}{2}t} - 3e^{-3t}$$

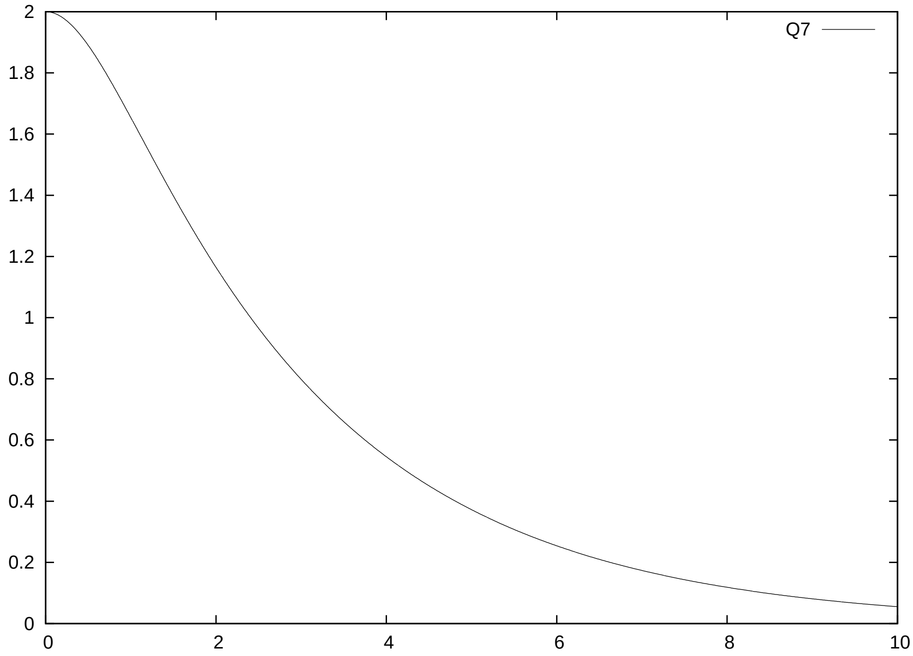
$$\begin{aligned} \phi(0) &= A + B + 1 &&= 2 \\ \dot{\phi}(0) &= \frac{-3+\sqrt{5}}{2}A + \frac{-3-\sqrt{5}}{2}B - 3 &&= 0 \end{aligned}$$

Solving the simultaneous equations gives $A = \frac{\sqrt{5}+9}{2\sqrt{5}}$ and $B = \frac{\sqrt{5}-9}{2\sqrt{5}}$ so the Full General Solution is

(FGS.7) $$\phi(t) = \frac{\sqrt{5}+9}{2\sqrt{5}}e^{\frac{-3+\sqrt{5}}{2}t} + \frac{\sqrt{5}-9}{2\sqrt{5}}e^{\frac{-3-\sqrt{5}}{2}t} + e^{-3t}$$

Damping ratio and amplitude of oscillation

The Auxiliary Equation (Aux.7) has two real roots so the system is over damped ($\zeta > 1$) and will not oscillate.



Question 8

(Q.8) $$\ddot{\psi} + 4\dot{\psi} + \psi = \cos 2t$$

(IC.8a) $$\psi(0) = 0$$

(IC.8b) $$\dot{\psi}(0) = 0$$

Particular Integral

$$\begin{aligned} &\psi_{PI}(t) = C\cos 2t + D\sin 2t \\ \implies &\dot{\psi}_{PI}(t) = -2C\sin 2t + 2D\cos 2t \\ \implies &\ddot{\psi}_{PI}(t) = -4C\cos 2t - 4D\sin 2t \end{aligned}$$

Find C and D by substituting $\psi_{PI}(t)$ and its derivatives into equation (Q.8) and equating coefficients of $\cos 2t$ and $\sin 2t$

(10) $$1 \times (-4C\cos 2t - 4D\sin 2t) + 4 \times (-2C\sin 2t + 2D\cos 2t) + 1 \times (C\cos 2t + D\sin 2t) = \cos 2t$$

$$\begin{aligned} \cos 2t &: -4C + 8D + C = 1 \\ \sin 2t &: -4D - 8C + D = 0 \end{aligned}$$

Solving the simultaneous equations gives $C = \frac{-3}{73}$ and $D = \frac{8}{73}$

(PI.8) $$\psi_{PI}(t) = -\frac{3}{73}\cos 2t + \frac{8}{73}\sin 2t$$

Complementary Function

(Aux.8) $$\lambda^2 + 4\lambda + 1 = 0$$

(Roots.8) $$\lambda = -2 \pm \sqrt{3}$$

There are two real roots to the Auxiliary Equation (Aux.8) so the Complementary Function is of the form

(CF.8) $$\psi_{CF}(t) = Ae^{(-2+\sqrt{3})t} + Be^{(-2-\sqrt{3})t}$$

General Solution

$$\begin{aligned} \psi(t) &= \psi_{CF}(t) + \psi_{PI}(t) \\ &= Ae^{(-2+\sqrt{3})t} + Be^{(-2-\sqrt{3})t} - \frac{3}{73}\cos 2t + \frac{8}{73}\sin 2t \end{aligned} \quad \text{(GS.8)}$$

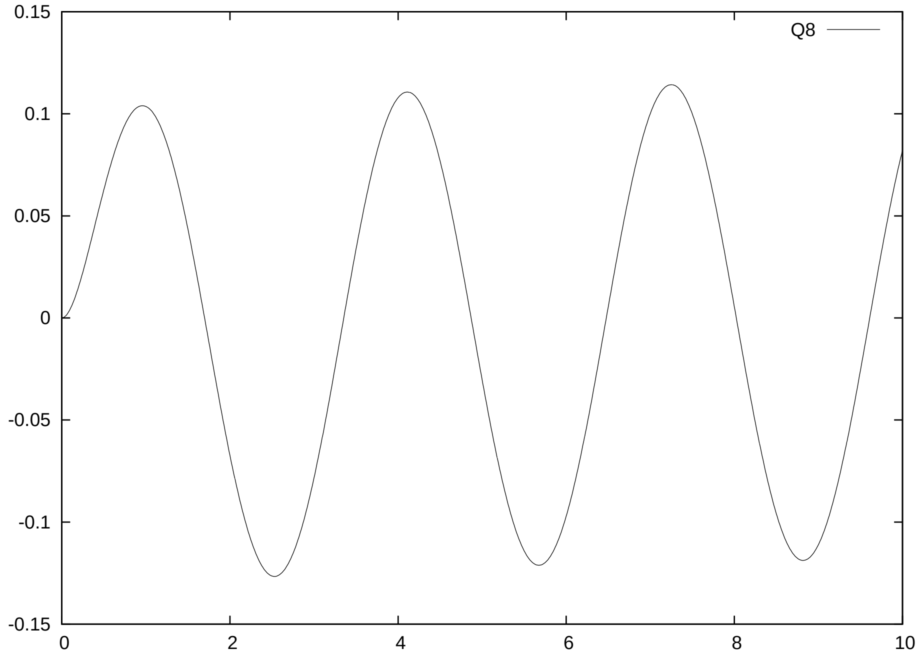
Coefficients A and B are found by applying the initial conditions to the General Solution and its derivative

(GS':8) $$\dot{\psi}(t) = (-2+\sqrt{3})Ae^{(-2+\sqrt{3})t} + (-2-\sqrt{3})Be^{(-2-\sqrt{3})t} + \frac{6}{73}\sin 2t + \frac{16}{73}\cos 2t$$

$$\begin{aligned} \psi(0) = A + B - \frac{3}{73} &= 0 \\ \dot{\psi}(0) = (-2+\sqrt{3})A + (-2-\sqrt{3})B + \frac{16}{73} &= 0 \end{aligned}$$

Solving the simultaneous equations gives $A = \frac{3\sqrt{3}-10}{146\sqrt{3}}$ and $B = \frac{3\sqrt{3}+10}{146\sqrt{3}}$

(FGS:8) $$\psi(t) = \frac{3\sqrt{3}-10}{146\sqrt{3}}e^{(-2+\sqrt{3})t} + \frac{3\sqrt{3}+10}{146\sqrt{3}}e^{(-2-\sqrt{3})t} - \frac{3}{73}\cos 2t + \frac{8}{73}\sin 2t$$



Damping ratio and amplitude of oscillation

The Auxiliary Equation (Aux.8) has two real roots so the system is over damped ($\zeta > 1$) and will not oscillate as a result of its free dynamics. However, the forcing function is oscillatory ($\cos 2t$) so the system will oscillate with a frequency of 2 [rad/s] (period $T = \frac{2\pi}{2} = \pi$ [s]). The amplitude of this forced oscillation is $\sqrt{\frac{3}{73}^2 + \frac{8}{73}^2}$.