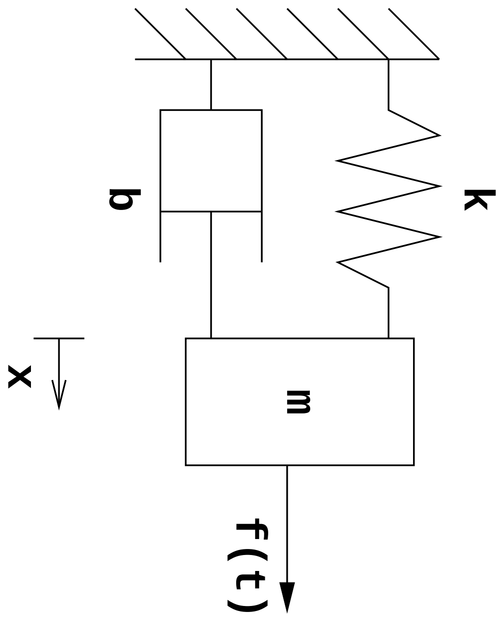
Mass-spring-damper



$$\ddot{x} + 4\dot{x} + 3x = mg$$

Parameters

$m = 1$ kg

$b = 4$ N s/m

$k = 3$ N/m

Forcing function

$f(t) = mg$ N

Initial conditions

$x(0) = 0$ m

$\dot{x}(0) = 0$ m/s

Auxiliary Equation:

$$\lambda^2 + 4\lambda + 3 = 0$$

$$\implies \lambda = \frac{-4 \pm \sqrt{4^2 - 4 \times 1 \times 3}}{2 \times 1} = -2 \pm 1$$

Complementary Function:

$$x_{cf}(t) = Ae^{-t} + Be^{-3t}$$

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Particular Integral:

$$x_{pi}(t) = C \qquad \Longrightarrow \quad \dot{x}_{pi}(t) = 0 \text{ and } \ddot{x}_{pi}(t) = 0$$

Substitute $x_{pi}(t)$ etc. into the differential equation

$$1 \times 0 + 4 \times 0 + 3 \times C = 1 \times g \qquad \Longrightarrow \quad C = g/3$$

General Solution

$$x(t) = x_{cf}(t) + x_{pi}(t) \quad = Ae^{-t} + Be^{-3t} + g/3$$

Apply initial conditions

$$\dot{x}(t) = -Ae^{-t} - 3Be^{-3t}$$

$$\dot{x}(0) = 0 = -A - 3B \qquad \Longrightarrow \quad A = -3B$$

$$x(0) = 0 = A + B + g/3$$

$$= 0 = -2B + g/3 \qquad \Longrightarrow \quad B = g/6$$

$$x(t) = -\frac{g}{2}e^{-t} + \frac{g}{6}e^{-3t} + \frac{g}{3}$$

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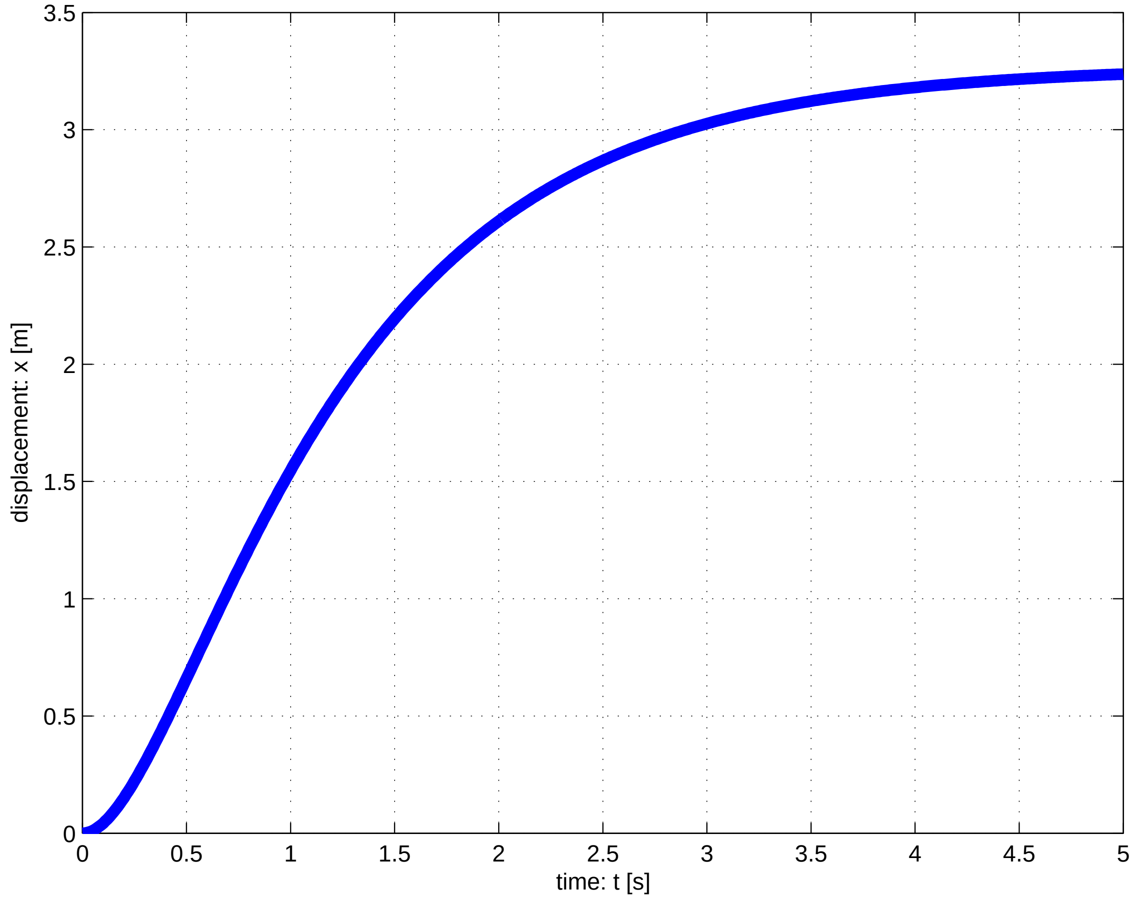
$$x(t) = -\frac{g}{2}e^{-t} + \frac{g}{6}e^{-3t} + \frac{g}{3}$$



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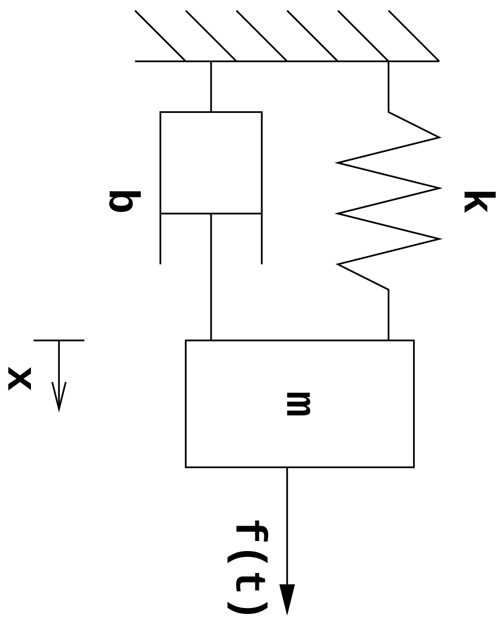

$$\ddot{x} + 4\dot{x} + 4x = mg$$

Parameters

$m = 1$ kg

$b = 4$ N s/m

$k = 4$ N/m

Forcing function

$f(t) = mg$ N

Initial conditions

$x(0) = 0$ m

$\dot{x}(0) = 0$ m/s

Auxiliary Equation:

$$\lambda^2 + 4\lambda + 4 = 0$$

$$\implies \lambda = \frac{-4 \pm \sqrt{4^2 - 4 \times 1 \times 4}}{2 \times 1} = -2$$

Complementary Function:

$$x_{cf}(t) = Ae^{-2t} + Bte^{-2t}$$

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Particular Integral:

$$x_{pi}(t) = C \qquad \Longrightarrow \quad \dot{x}_{pi}(t) = 0 \text{ and } \ddot{x}_{pi}(t) = 0$$

Substitute $x_{pi}(t)$ etc. into the differential equation

$$1 \times 0 + 4 \times 0 + 4 \times C = 1 \times g \qquad \Longrightarrow \quad C = g/4$$

General Solution

$$x(t) = x_{cf}(t) + x_{pi}(t) \quad = Ae^{-2t} + Bte^{-2t} + g/4$$

Apply initial conditions

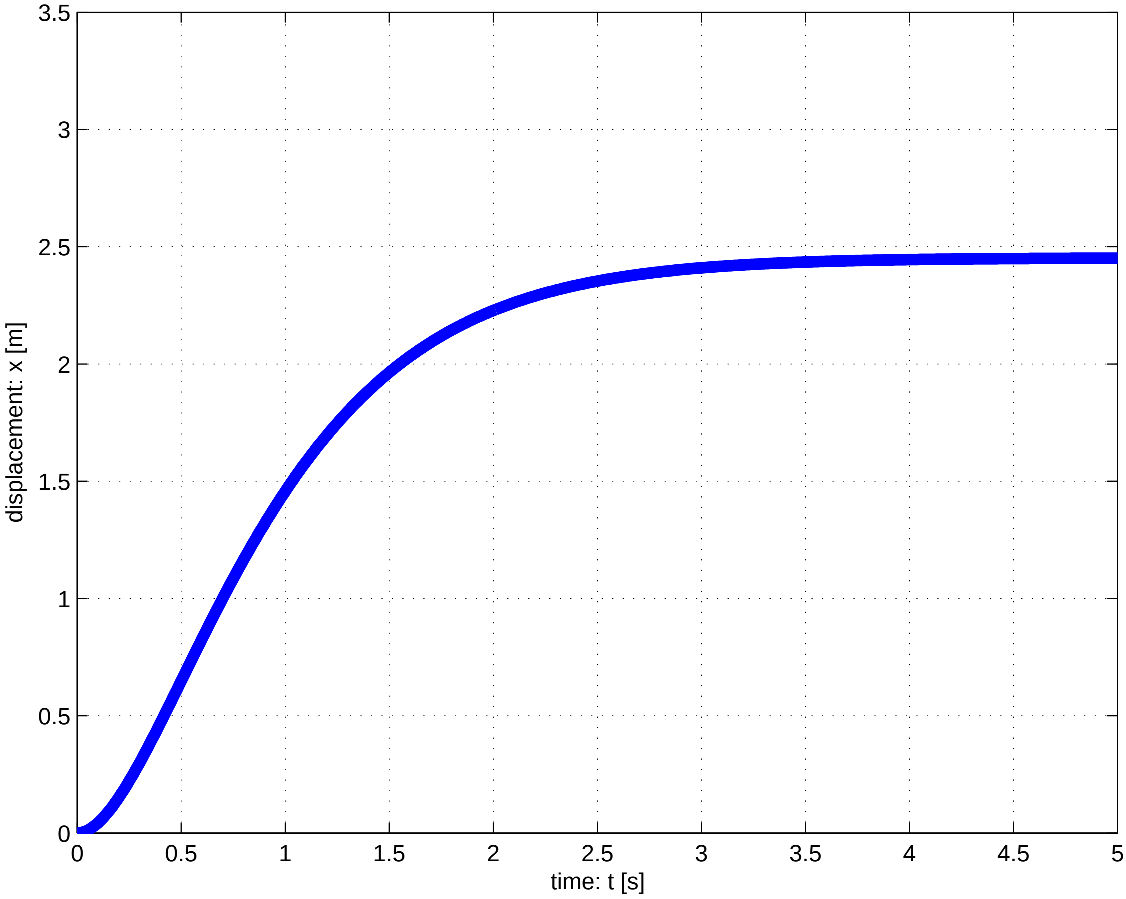
$$x(0) = 0 = A + g/4 \qquad \Longrightarrow \quad A = -g/4$$

$$\dot{x}(t) = -2Ae^{-2t} + Be^{-2t} - 2Bte^{-2t}$$

$$\dot{x}(0) = 0 = -2A + B \qquad \Longrightarrow \quad B = 2A$$

$$x(t) = -\frac{g}{4}e^{-2t} - \frac{g}{2}te^{-2t} + \frac{g}{4}$$

$$x(t) = -\frac{g}{4}e^{-2t} - \frac{g}{2}te^{-2t} + \frac{g}{4}$$



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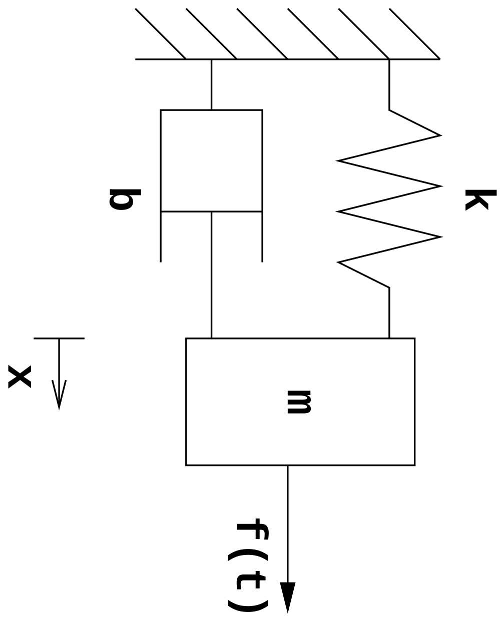

$$\ddot{x} + 4\dot{x} + 5x = mg$$

Parameters

$m = 1$ kg

$b = 4$ N s/m

$k = 5$ N/m

Forcing function

$f(t) = mg$ N

Initial conditions

$x(0) = 0$ m

$\dot{x}(0) = 0$ m/s

Auxiliary Equation:

$$\lambda^2 + 4\lambda + 5 = 0$$

$$\implies \lambda = \frac{-4 \pm \sqrt{4^2 - 4 \times 1 \times 5}}{2 \times 1} = -2 \pm j$$

Complementary Function:

$$x_{cf}(t) = e^{-2t}(A \sin t + B \cos t)$$

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Particular Integral:

$$x_{pi}(t) = C \qquad \Longrightarrow \quad \dot{x}_{pi}(t) = 0 \text{ and } \ddot{x}_{pi}(t) = 0$$

Substitute $x_{pi}(t)$ etc. into the differential equation

$$1 \times 0 + 4 \times 0 + 5 \times C = 1 \times g \qquad \Longrightarrow \quad C = g/5$$

General Solution

$$x(t) = x_{cf}(t) + x_{pi}(t) \quad = e^{-2t}(A \sin t + B \cos t) + g/5$$

Apply initial conditions

$$x(0) = 0 = B + g/5 \qquad \Longrightarrow \quad B = -g/5$$

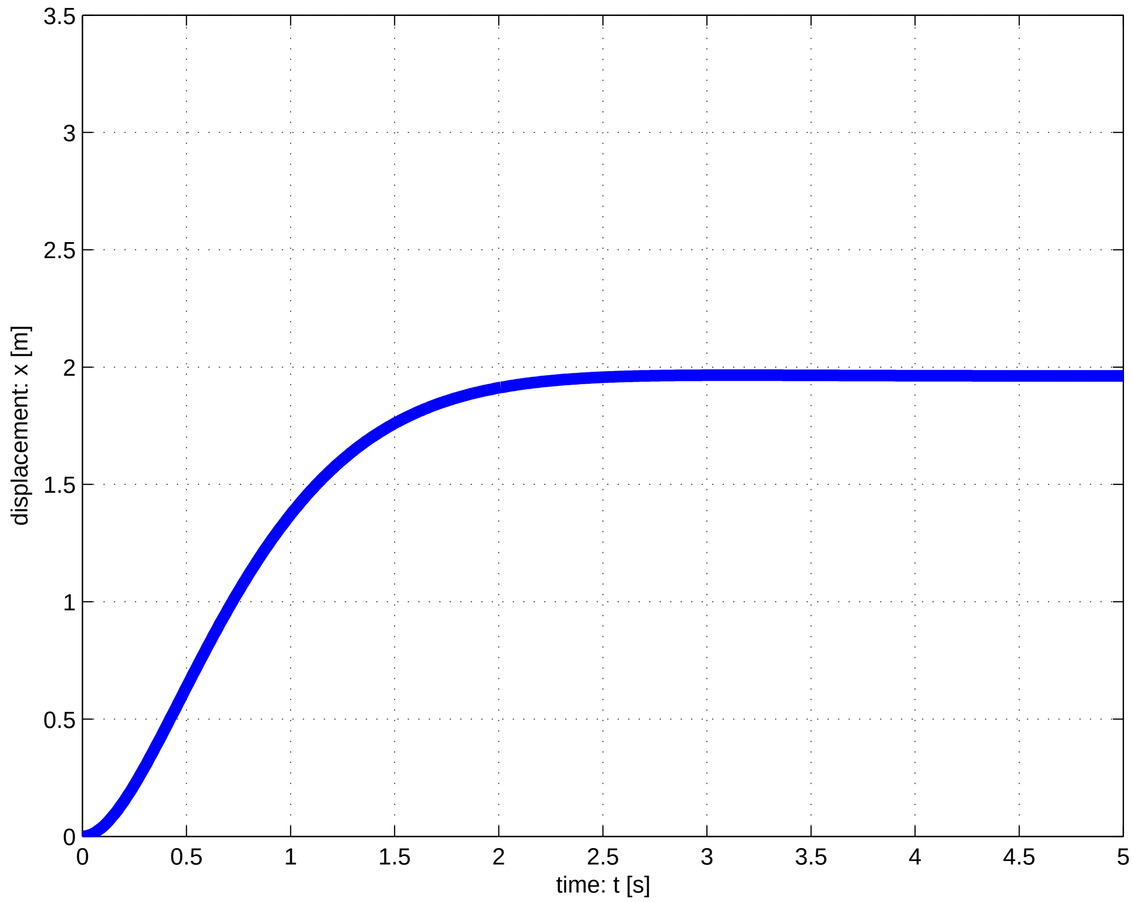
$$\dot{x}(t) = -2e^{-2t}(A \sin t + B \cos t) + e^{-2t}(A \cos t - B \sin t)$$

$$\dot{x}(0) = 0 = -2B + A \qquad \Longrightarrow \quad A = 2B$$

$$x(t) = e^{-2t}\left(-\frac{2g}{5} \sin t - \frac{g}{5} \cos t\right) + \frac{g}{5}$$

Mass-spring damper example 3 graphical solution

$$x(t) = e^{-2t}\left(-\frac{2g}{5}\sin t - \frac{g}{5}\cos t\right) + \frac{g}{5}$$



Write the form of Particular Integral to solve

$$a\ddot{x} + b\dot{x} + cx = f(t)$$

where

$$f(t) = 7t + 1$$

$$f(t) = -t^2$$

$$f(t) = mg$$

$$f(t) = -3\cos(2t)$$

$$f(t) = -2e^{-3t}$$

$$f(t) = -9t^2 + 4e^{-5t} - e^t$$

Answers:

$$x_{pi} = Ct + D$$

$$x_{pi} = Ct^2 + Dt + E$$

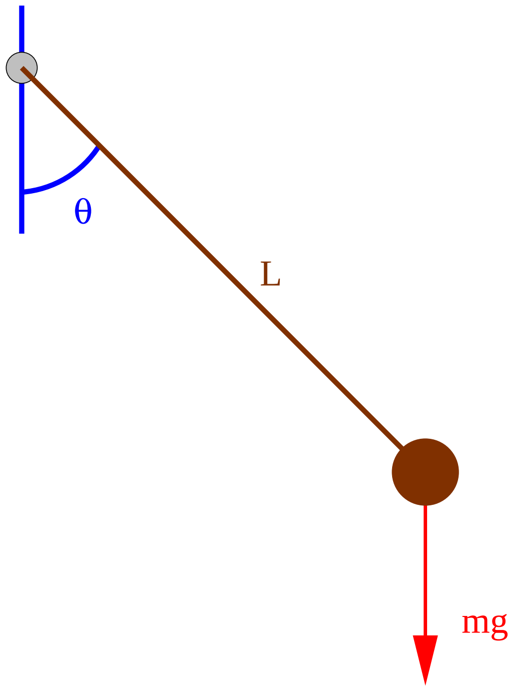
$$x_{pi} = C$$

$$x_{pi} = C\sin(2t) + D\cos(2t)$$

$$x_{pi} = Ce^{-3t}$$

$$x_{pi} = Ct^2 + Dt + E + Fe^{-5t} + Ge^t$$

An undamped, ideal pendulum released from rest



$$ml^2\ddot{\theta} = -mgl\sin\theta$$

$$\sin\theta \approx \theta \qquad \forall|\theta| << 1$$

$$\ddot{\theta} + \frac{g}{l}\theta = 0$$

Initial conditions:

$\theta(0) = \theta_0$ rad

$\dot{\theta}(0) = 0$ rad/s

Auxiliary Equation:

$$\lambda^2 + g/l = 0$$

Complementary Function:

$$\theta_{cf}(t) = e^{0t}\left(A\sin\left(\sqrt{\frac{g}{l}}t\right) + B\cos\left(\sqrt{\frac{g}{l}}t\right)\right)$$

Particular Integral is zero, hence General Solution:

$$\theta(t) = \theta_0\cos(\omega_n t)$$

where $\omega_n = \sqrt{g/l}$

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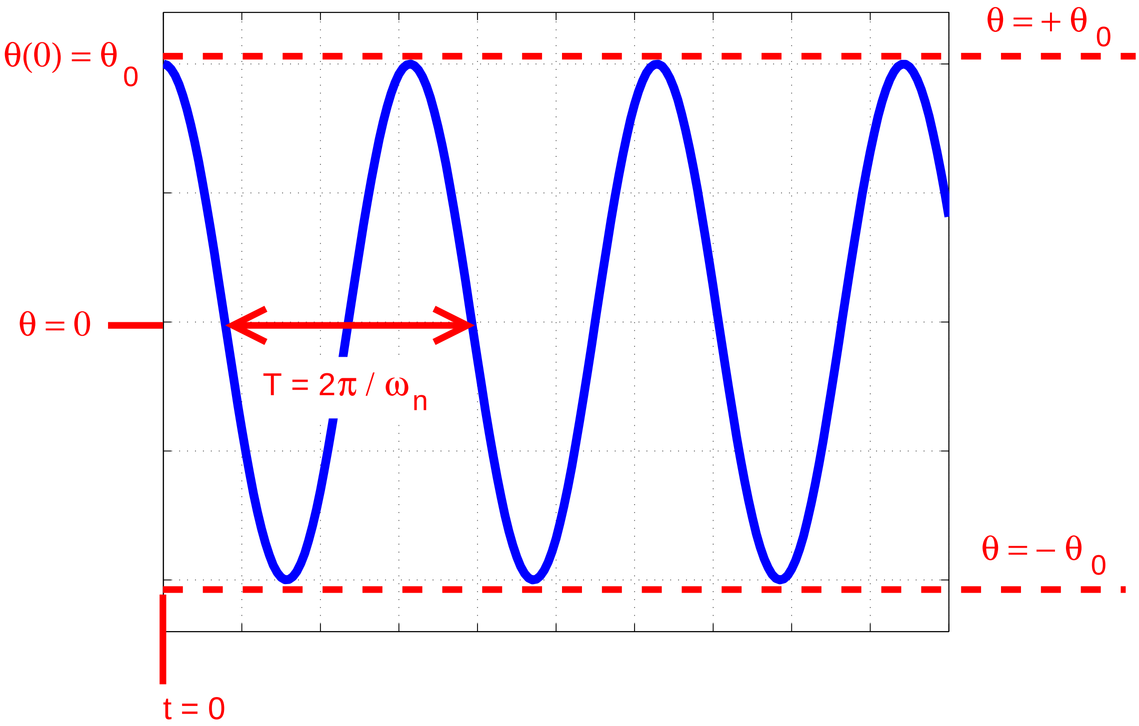

- The curve starts at $(0, \theta_0)$
- The pendulum oscillates, with amplitude θ_0, about the equilibrium point $\theta = 0$ and the amplitude does not decay or grow with time
- The period of the oscillation is $T = 2\pi/\omega_n\,\mathrm{s} = 2\pi\sqrt{l/g}\,\mathrm{s}$

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Change the initial conditions so that the pendulum does not start from rest

$$\ddot{\theta} + \omega_n^2 \theta = 0$$

$$\theta(0) = \theta_0 \text{ and } \dot{\theta}(0) = \dot{\theta}_0$$

Complementary Function:

$$\theta_{cf}(t) = (A \sin(\omega_n t) + B \cos(\omega_n t))$$

Recall the trigonometric identity

$$\sin\alpha \sin\beta + \cos\alpha \cos\beta \equiv cos(\alpha - \beta)$$

Let

$$\frac{A}{\sqrt{A^2+B^2}} = \sin\phi \text{ and } \frac{B}{\sqrt{A^2+B^2}} = \cos\phi$$

Then the Complementary Function is

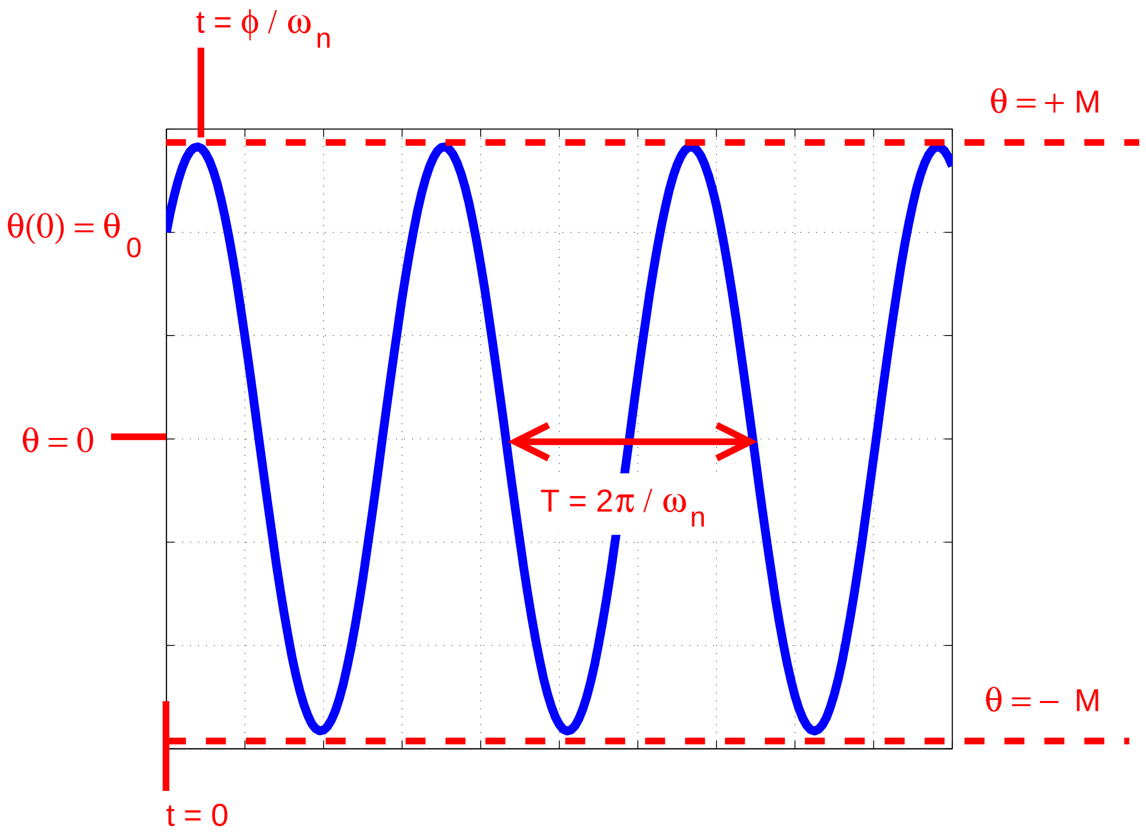
$$\theta_{cf}(t) = M \cos(\omega_n t - \phi)$$

where $M = \sqrt{A^2 + B^2}$ and $\phi = \arctan(A/B)$.

The pendulum is unforced so the Particular Integral is zero. Substituting in the initial conditions, the General Solution is

$$\begin{aligned}\theta(t) &= \frac{\dot{\theta}_0}{\omega_n}\sin(\omega_n t) + \theta_0\cos(\omega_n t)\\ &= M\cos(\omega_n t - \phi)\end{aligned}$$

where $M = \sqrt{\left(\frac{\dot{\theta}_0}{\omega_n}\right)^2 + \theta_0^2}$ and $\phi = \arctan\frac{\dot{\theta}_0/\omega_n}{\theta_0}$.



Suppose that friction is added to the system

$$\ddot{\theta} + \mu\dot{\theta} + \omega_n^2\theta = 0$$

Auxiliary equation

$$\lambda = \frac{-\mu \pm \sqrt{\mu^2 - 4\omega_n^2}}{2}$$

If $\mu < 2\omega_n$ then

$$\lambda = -\frac{\mu}{2} \pm j\frac{\sqrt{4\omega_n^2 - \mu^2}}{2}$$

Define a Damping Ratio

$$\zeta = \frac{\mu}{2\omega_n}$$

The solution to the Auxiliary Equation is then

$$\lambda = -\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2}$$

And the Complementary Function is

$$\begin{aligned}\theta(t) &= e^{-\zeta\omega_n t}(A\sin\omega t + B\cos\omega t)\\ &= Me^{-\zeta\omega_n t}\cos(\omega t - \phi)\end{aligned}$$

where $\omega = \omega_n\sqrt{1-\zeta^2}$

$$\boxed{\ddot{\theta} + 2\zeta\omega_n\dot{\theta} + \omega_n^2\theta = 0}$$

if $\zeta > 1$ the system is *over-damped*

$$\theta_{cf}(t) = Ae^{\lambda_1 t} + Be^{\lambda_2 t}$$

if $\zeta = 1$ the system is *critically-damped*

$$\theta_{cf}(t) = (A + Bt)e^{\lambda t}$$

if $0 < \zeta < 1$ the system is *under-damped*

$$\theta_{cf}(t) = Me^{-\zeta\omega_n t}\cos(\omega t - \phi)$$

if $\zeta = 0$ the system is *undamped*

$$\theta_{cf}(t) = M\cos(\omega t - \phi)$$

where

$$\omega = \omega_n\sqrt{1 - \zeta^2}$$

$$\phi = \arctan\frac{A}{B}$$

$$M = \sqrt{A^2 + B^2}$$

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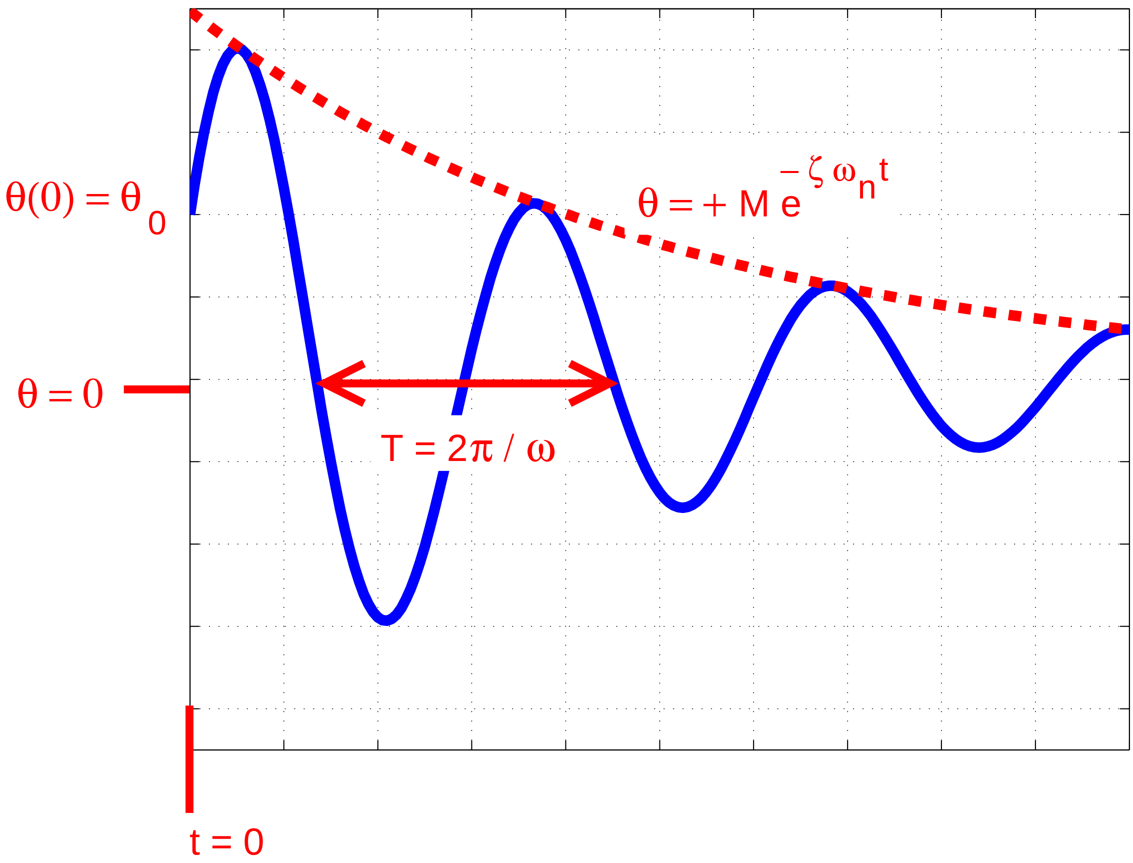
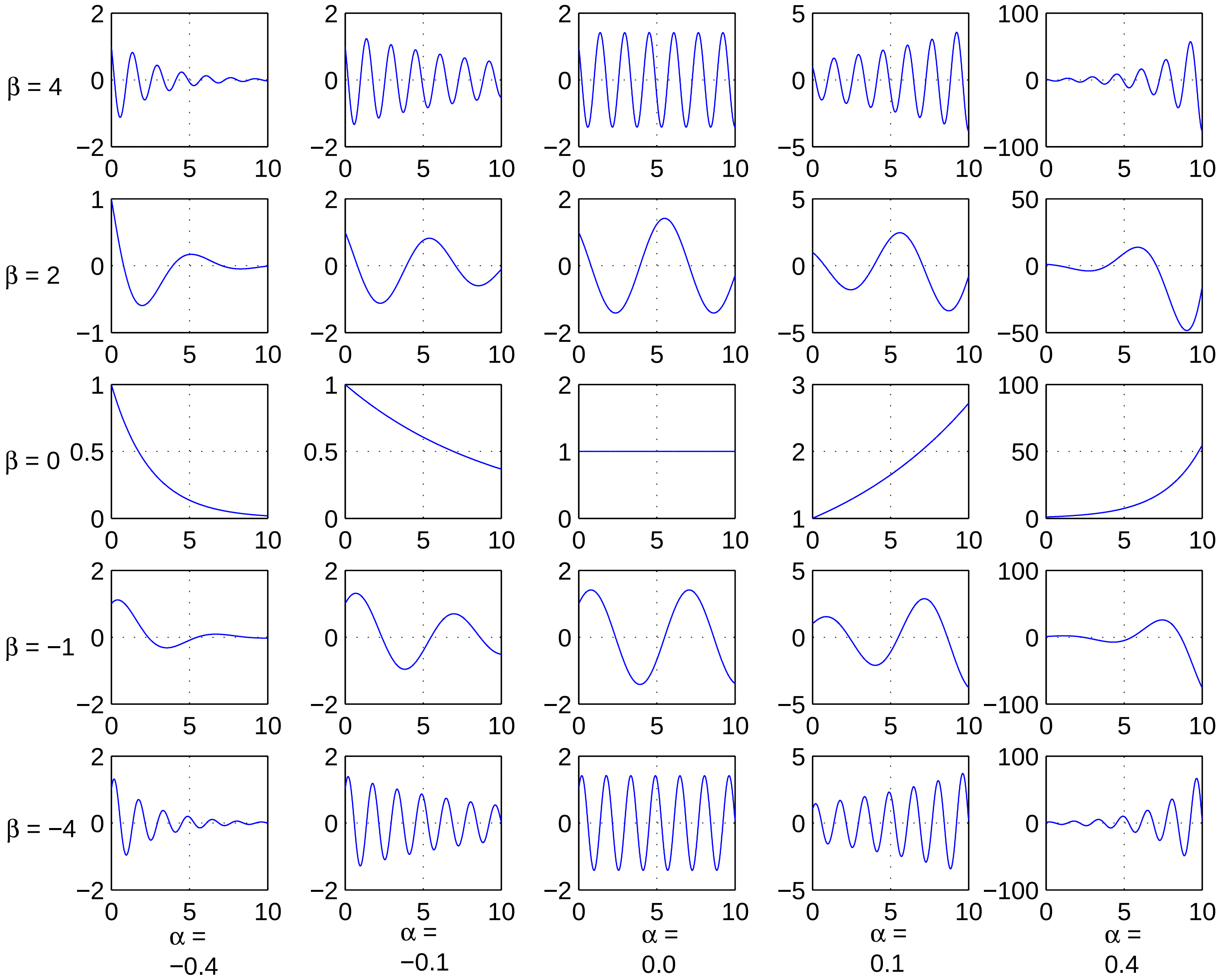
For a free (i.e. unforced) under-damped pendulum the General Solution is

$$\theta(t) = Me^{-\zeta\omega_n t}\cos(\omega t - \phi)$$



$e^{\alpha t}(\sin \beta t + \cos \beta t)$



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Consider an undamped oscillating system with a sinusoidal input function

$$\ddot{x} + \omega_n^2 x = U \sin \Omega t$$

The Complementary Function is

$$\begin{aligned} x_{cf}(t) &= e^{0t}\left(A\sin(\omega_n t) + B\cos(\omega_n t)\right) \\ &= M\cos(\omega_n t - \phi) \end{aligned}$$

where $M = \sqrt{A^2 + B^2}$ and $\phi = \arctan \frac{A}{B}$.

For the Particular Integral, try

$$\begin{aligned} x_{pi}(t) &= C\sin(\Omega t) + D\cos(\Omega t) \\ \implies \dot{x}_{pi} &= C\Omega\cos(\Omega t) - D\Omega\sin(\Omega t) \\ \ddot{x}_{pi} &= -C\Omega^2\sin(\Omega t) - D\Omega^2\cos(\Omega t) \end{aligned}$$

Substitute the PI into the ODE and equate like terms

$$\begin{aligned} \sin(\Omega t) : &\qquad C(\omega_n^2 - \Omega^2) = U \\ \cos(\Omega t) : &\qquad D(\omega_n^2 - \Omega^2) = 0 \end{aligned}$$

$$\implies D = 0 \text{ and } C = \frac{U}{\omega_n^2 - \Omega^2}$$

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The General Solution

$$x(t) = M\cos(\omega_n t - \phi) + \frac{U\sin(\Omega t)}{\omega_n^2 - \Omega^2}$$

indicates that the response will be a combination of two oscillations. However, as the driving frequency Ω approaches the system natural frequency ω_n, the amplitude of the second term will increase. When $\Omega = \omega_n$, the displacement is infinite.

Try a different Particular Integral

$$x_{pi} = t\left(C\sin(\Omega t) + D\cos(\Omega t)\right)$$
$$\dot{x}_{pi} = (C - tD\Omega)\sin(\Omega t) + (D + tC\Omega)\cos(\Omega t)$$
$$\ddot{x}_{pi} = (-2D\Omega - tC\Omega^2)\sin(\Omega t) + (2C\Omega - tD\Omega^2)\cos(\Omega t)$$

Substituting into the ODE and equating like terms

$$\begin{aligned} \sin(\Omega t): \quad & -2D\Omega + tC(\omega_n^2 - \Omega^2) &= U \\ \cos(\Omega t): \quad & +2C\Omega + tD(\omega_n^2 - \Omega^2) &= 0 \end{aligned}$$

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Now, if $\Omega = \omega_n$ then $(\omega_n^2 - \Omega^2) = 0$, hence

$$\begin{aligned} \sin(\omega_n t): \quad & -2D\omega_n = U \\ \cos(\omega_n t): \quad & +2C\omega_n = 0 \end{aligned}$$

$$\implies C = 0 \text{ and } D = -\frac{U}{2\omega_n}$$

Thus the General Solution is

$$x(t) = M\cos(\omega_n t - \phi) - \frac{U}{2\omega_n} t \sin(\omega_n t)$$

and the amplitude of the oscillation increases proportionally with time.

If damping is added to the system

$$\ddot{x} + \mu\dot{x} + \omega_n^2 x = U\sin(\Omega t)$$

then there is no singularity problem with the Particular Integral. Substituting $x_{pi} = C\sin(\Omega t) + D\cos(\Omega t)$ and its derivatives into the ODE, gives

$$C = \frac{U(\omega_n^2 - \Omega^2)}{(\omega_n^2 - \Omega^2)^2 + (\mu\Omega)^2} \text{ and } D = \frac{-U\mu\Omega}{(\omega_n^2 - \Omega^2)^2 + (\mu\Omega)^2}$$

Let $N\sin\gamma = C/U$ and $N\cos\gamma = D/U$. It follows that the magnitude of the forced oscillation term is

$$UN = \sqrt{C^2 + D^2} = U\sqrt{\frac{1}{(\omega_n^2 - \Omega^2)^2 + (\mu\Omega)^2}}$$

and its associated phase lag is

$$\gamma = \arctan\frac{C}{D} = \arctan\frac{\omega_n^2 - \Omega^2}{-\mu\Omega}$$

The General Solution is then

$$x(t) = Me^{-\zeta\omega_n t}\cos(\omega t - \phi) + UN\cos(\Omega t - \gamma)$$

As in the undamped case, maximum transfer of energy occurs if the forcing frequency is equal to the natural frequency, but the presence of damping prevents infinite oscillations.

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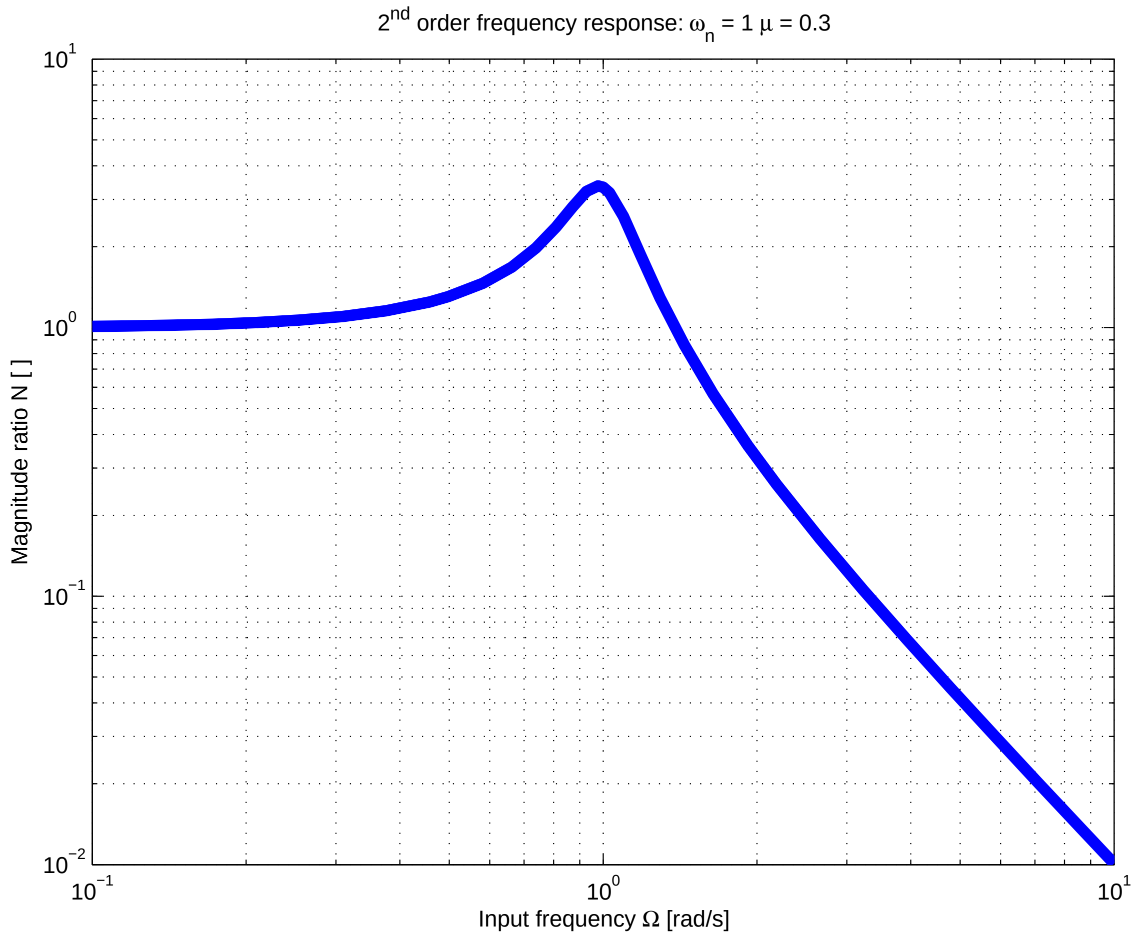
Bode Diagram:

$$\ddot{x} + \mu\dot{x} + \omega_n^2 = \sin(\Omega t)$$

$$N = \sqrt{\frac{1}{(\omega_n^2 - \Omega^2)^2 + (\mu\Omega)^2}}$$

$$\mu = 0.3, \omega_n = 1$$

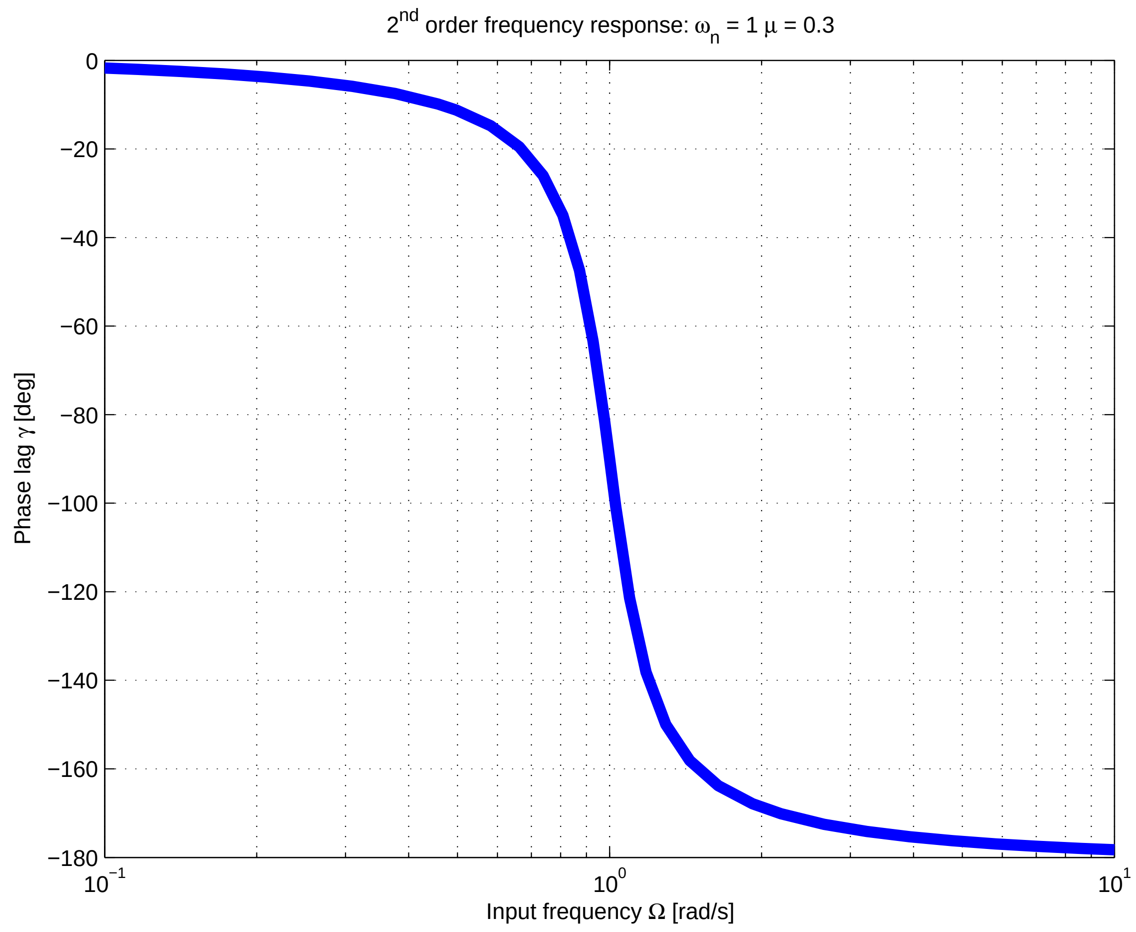


Bode Diagram:

$$\ddot{x} + \mu\dot{x} + \omega_n^2 = \sin(\Omega t)$$

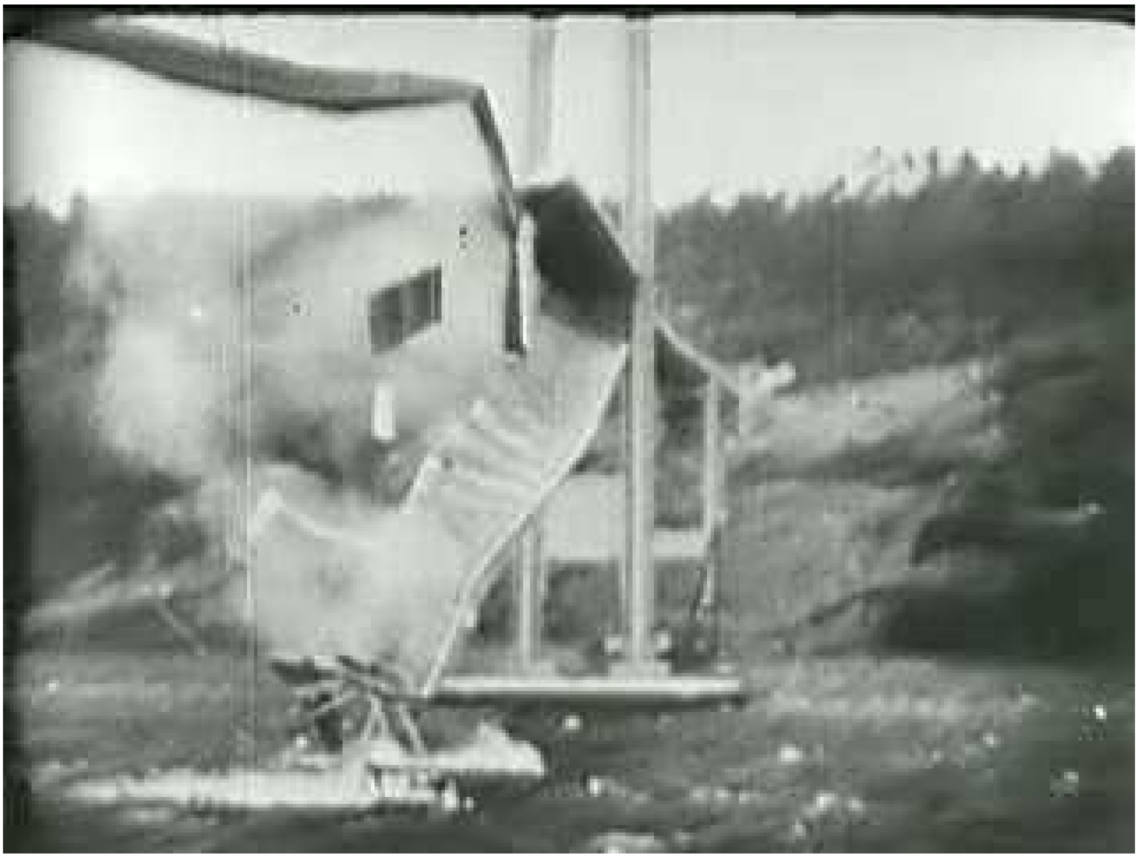
$$\gamma = \arctan\frac{\omega_n^2 - \Omega^2}{-\mu\Omega}$$

$$\mu = 0.3, \omega_n = 1$$



Tacoma Narrows Bridge film:

http://www.archive.org/details/Pa2096Tacoma



http://en.wikipedia.org/wiki/Image:Tacoma_Narrows_Bridge_Falling.png

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To solve a linear homogeneous differential equation, i.e.

$$a_n \frac{d^n x}{dt^n} + \cdots + a_2 \frac{d^2 x}{dt^2} + a_1 \frac{dx}{dt} + a_0 x = f(t)$$

Form the Auxiliary (Characteristic) Equation

$$a_n \lambda^n + \cdots + a_2 \lambda^2 + a_1 \lambda + a_0 = 0$$

Find values of λ by solving the Auxiliary equation and hence form the Complementary Function

1^{st} order:

$$x_{cf}(t) = e^{\lambda t}$$

2^{nd} order:

$$x_{cf}(t) = \begin{cases} Ae^{\lambda_1 t} + Be^{\lambda_2 t} & : (\lambda_1 \neq \lambda_2) \\ (At + B)e^{\lambda t} & : (\lambda_1 = \lambda_2) \\ e^{\alpha t}\left(A \sin(\beta t) + B \cos(\beta t)\right) & : \lambda = \alpha \pm j\beta \end{cases}$$

For higher orders, factorise down to 1^{st} and 2^{nd} order terms and then sum the resulting expressions.

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Choose a form of Particular Integral that is similar to the RHS and its derivatives. Differentiate n times with respect to t and substitute the expressions into the original equation

$$a_n \frac{d^n x_{pi}}{dt^n} + \cdots + a_2 \frac{d^2 x_{pi}}{dt^2} + a_1 \frac{dx_{pi}}{dt} + a_0 x_{pi} = f(t)$$

Then equate coefficients on the LHS and RHS to completely specify the Particular Integral.

Add the Complementary Function to the Particular Integral to form the General Solution.

$$x(t) = x_{cf}(t) + x_{pi}(t)$$

Use the boundary conditions to determine the remaining coefficients from the Complementary Function.

$$x(0) = x_{cf}(0) + x_{pi}(0) = x_0$$
$$\dot{x}(0) = \dot{x}_{cf}(0) + \dot{x}_{pi}(0) = \dot{x}_0$$

&c.

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