

# Applicable Maths 1A : Matrices 2009/10

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April 15, 2010

assessment: 2 exam questions (out of 10)

tuition:  $\leq$  10 lectures + tutorials

## **“What advice would you give to someone taking the course next year?”**

What last year’s students said:

- “Matrices were interesting” (x4)
- “Keep on top of work from the start”
- “Go to lectures”
- “Always listen to the lecturer even if you miss note-taking”
- “Advise new students to keep notes organised”
- “Re-read notes every evening”
- “Go to tutorials even if you did Advanced Higher Maths”
- “100% test 2 hour class good” (x2)

**NB** These are new notes. There will be errors. Please check any results that you are unsure of.

# Notation

## What is a matrix?

### Some definitions

- Dimensions: An  $m \times n$  matrix has  $m$  rows and  $n$  columns.

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} \quad \text{is a } 2 \times 3 \text{ matrix}$$

- $a_{ij}$  is the element in the  $i^{\text{th}}$  row and  $j^{\text{th}}$  column of matrix  $A$

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix}; \quad \text{e.g. } a_{22} = 5 \text{ (using } A \text{ above)}$$

- A *row vector* has a single row ( $m = 1$ )

$$R = (r_{11} \quad r_{12} \quad r_{13}) \quad \text{is a row vector}$$

- A *column vector* has a single column ( $n = 1$ )

$$C = \begin{pmatrix} c_{11} \\ c_{21} \\ c_{31} \end{pmatrix} = (c_{11} \quad c_{21} \quad c_{31})^T \quad \text{is a column vector}$$

(The superscript  $T$  is the *transpose* operator)

- A *square matrix* has the same number of rows and columns ( $m = n$ )

$$S = \begin{pmatrix} s_{11} & s_{12} & s_{13} \\ s_{21} & s_{22} & s_{23} \\ s_{31} & s_{32} & s_{33} \end{pmatrix} \quad \text{is a square matrix}$$

- The *trace* of a square matrix ( $\text{tr}(A)$ ) is the sum of the elements in the leading diagonal

$$S = \begin{pmatrix} \mathbf{s}_{11} & s_{12} & s_{13} \\ s_{21} & \mathbf{s}_{22} & s_{23} \\ s_{31} & s_{32} & \mathbf{s}_{33} \end{pmatrix}; \quad \text{tr}(S) = \sum_{i=1}^m s_{ii} = s_{11} + s_{22} + s_{33}$$

- A *lower triangular* matrix is zero above the leading diagonal

$$L = \begin{pmatrix} \mathbf{l}_{11} & 0 & 0 \\ l_{21} & \mathbf{l}_{22} & 0 \\ l_{31} & l_{32} & \mathbf{l}_{33} \end{pmatrix} \quad \text{is lower triangular}$$

- An *upper triangular* matrix is zero below the leading diagonal

$$U = \begin{pmatrix} \mathbf{u}_{11} & u_{12} & u_{13} \\ 0 & \mathbf{u}_{22} & u_{23} \\ 0 & 0 & \mathbf{u}_{33} \end{pmatrix} \quad \text{is upper triangular}$$

- A *diagonal* matrix is zero off the leading diagonal

$$D = \begin{pmatrix} \mathbf{d_{11}} & 0 & 0 \\ 0 & \mathbf{d_{22}} & 0 \\ 0 & 0 & \mathbf{d_{33}} \end{pmatrix} \quad \text{is a diagonal matrix}$$

- The *Identity* matrix  $I$  is a diagonal matrix with every element unity on the leading diagonal

$$I_3 = \begin{pmatrix} \mathbf{1} & 0 & 0 \\ 0 & \mathbf{1} & 0 \\ 0 & 0 & \mathbf{1} \end{pmatrix}; \quad I_4 = \begin{pmatrix} \mathbf{1} & 0 & 0 & 0 \\ 0 & \mathbf{1} & 0 & 0 \\ 0 & 0 & \mathbf{1} & 0 \\ 0 & 0 & 0 & \mathbf{1} \end{pmatrix} \quad \text{are Identity matrices}$$

### Transpose and symmetry

- The *transpose*  $A^T$  is a matrix formed by flipping  $A$  about its leading diagonal.

$$A = \begin{pmatrix} \mathbf{1} & \mathbf{2} & \mathbf{3} \\ \mathbf{4} & \mathbf{5} & \mathbf{6} \end{pmatrix}; \quad A^T = \begin{pmatrix} \mathbf{1} & \mathbf{4} \\ \mathbf{2} & \mathbf{5} \\ \mathbf{3} & \mathbf{6} \end{pmatrix} \quad a_{ij} \rightarrow a_{ji} \quad \forall i \in [1, m], j \in [1, n]$$

- A *symmetric* matrix  $A$  is a square matrix that is equal to its transpose  $A = A^T$

$$A = \begin{pmatrix} \mathbf{1} & \mathbf{2} & \mathbf{3} \\ \mathbf{2} & \mathbf{4} & \mathbf{5} \\ \mathbf{3} & \mathbf{5} & \mathbf{6} \end{pmatrix} = A^T \quad \text{is symmetric: } a_{ij} = a_{ji} \quad \forall i, j \in [1, m]$$

equal to its transpose  $A^T$  if and only if (iff) the  $A$  is square and *symmetric*

- A *skew-symmetric* or *anti-symmetric* matrix  $A$  is a square matrix such that the sum of  $A$  and its transpose is zero, i.e.  $A = -A^T$

$$A = \begin{pmatrix} \mathbf{0} & \mathbf{2} & \mathbf{3} \\ \mathbf{-2} & \mathbf{0} & \mathbf{5} \\ \mathbf{-3} & \mathbf{-5} & \mathbf{0} \end{pmatrix}; \quad A^T = \begin{pmatrix} \mathbf{0} & \mathbf{-2} & \mathbf{-3} \\ \mathbf{2} & \mathbf{0} & \mathbf{-5} \\ \mathbf{3} & \mathbf{5} & \mathbf{0} \end{pmatrix}; \quad A = -A^T; a_{ij} = -a_{ji} \quad \forall i, j \in [1, m]$$

Note that the leading diagonal of an anti-symmetric matrix must be zero

## Matrix properties

### Equality

Two matrices  $A$  and  $B$  are *equal* if their dimensions are identical and they contain the same elements in the same positions

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{pmatrix}; B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{pmatrix} \quad \text{are equal iff } a_{ij} = b_{ij} \quad \forall i \in [1, m], j \in [1, n]$$

### Addition and subtraction

Two matrices  $A$  and  $B$  may be *added* (*subtracted*) if their dimensions are identical. Simply add (subtract) each element in  $B$  to (from) the corresponding element in  $A$

$$C = A + B = \begin{pmatrix} a_{11} + b_{11} & a_{12} + b_{12} \\ a_{21} + b_{21} & a_{22} + b_{22} \\ a_{31} + b_{31} & a_{32} + b_{32} \end{pmatrix} \quad c_{ij} = a_{ij} + b_{ij} \quad \forall i \in [1, m], j \in [1, n]$$

$$D = A - B = \begin{pmatrix} a_{11} - b_{11} & a_{12} - b_{12} \\ a_{21} - b_{21} & a_{22} - b_{22} \\ a_{31} - b_{31} & a_{32} - b_{32} \end{pmatrix} \quad d_{ij} = a_{ij} - b_{ij} \quad \forall i \in [1, m], j \in [1, n]$$

Examples:

$$\begin{aligned} A &= \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}; & A + B &= \begin{pmatrix} 8 & 10 & 12 \\ 4 & 6 & 8 \end{pmatrix} \\ B &= \begin{pmatrix} 7 & 8 & 9 \\ 0 & 1 & 2 \end{pmatrix}; & A - B &= \begin{pmatrix} -6 & -6 & -6 \\ -4 & 4 & 4 \end{pmatrix} \end{aligned}$$

### Multiplication by a scalar

A matrix  $A$  may be *multiplied* by a *scalar*  $\lambda$  in a similar manner, by multiplying each element of  $A$  by  $\lambda$

$$\lambda A = \begin{pmatrix} \lambda a_{11} & \lambda a_{12} \\ \lambda a_{21} & \lambda a_{22} \\ \lambda a_{31} & \lambda a_{32} \end{pmatrix}$$

Example:

$$A = \begin{pmatrix} 0 & 1 \\ 2 & 3 \\ 4 & 2 \end{pmatrix}; \quad \lambda = 2; \quad \lambda A = A\lambda = \begin{pmatrix} 0 & 2 \\ 4 & 6 \\ 8 & 2 \end{pmatrix}$$

## Rules of addition

Matrices are:

*commutative*

$$A + B = B + A$$

*associative*

$$(A + B) + C = A + (B + C)$$

*distributive*

$$\lambda(A + B) = \lambda A + \lambda B$$

## Further properties of the transpose

If  $A$  and  $B$  are two matrices with the same dimensions (and hence can be added), the transpose of the sum is equal to the sum of the transposes of each matrix

$$[A + B]^T = A^T + B^T$$

The transpose of the transpose of a matrix  $A^{TT}$  must yield the original matrix  $A$

$$A^{TT} = A$$

A matrix  $A$  may be added to its transpose  $A^T$  iff  $A$  is square. In this case, the result must be symmetric

$$[A + A^T]^T = A^T + A^{TT} = [A^T + A] = [A + A^T] \quad \implies \text{symmetric}$$

Similarly,

$$[A - A^T]^T = A^T - A^{TT} = [A^T - A] = -[A - A^T] \quad \implies \text{anti-symmetric}$$

Note that any square matrix can therefore be expressed as the sum of a symmetric and anti-symmetric matrix

$$A = \underbrace{\frac{1}{2}[A + A^T]}_{\text{symmetric}} + \underbrace{\frac{1}{2}[A - A^T]}_{\text{anti-symmetric}}$$

## Matrix multiplication: the matrix product

Two matrices  $A$  and  $B$  may be multiplied if the number of columns in  $A$  is the same as the number of rows in  $B$ . The resulting product will have the same number of rows as  $A$  and the same number of columns as  $B$ .

If  $A$  has dimensions  $(m \times p)$  and  $B$  has dimensions  $(p \times n)$ , then the product  $C = AB$  will have the dimensions  $(m \times n)$ .

The elements of the product  $C = AB$  are defined as

$$c_{ij} = \sum_{k=1}^p (a_{ik} \times b_{kj}) \quad \forall i \in [1, m], j \in [1, n]$$

Let  $C = AB$

$$\begin{pmatrix} a_{21} & a_{22} & a_{23} \end{pmatrix} \begin{pmatrix} b_{11} \\ b_{21} \\ b_{31} \end{pmatrix} = \begin{pmatrix} c_{21} \end{pmatrix}$$

$$c_{21} = a_{21} \times b_{11} + a_{22} \times b_{21} + a_{23} \times b_{31}$$

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \end{pmatrix} \begin{pmatrix} b_{13} \\ b_{23} \\ b_{33} \end{pmatrix} = \begin{pmatrix} c_{13} \end{pmatrix}$$

$$c_{13} = a_{11} \times b_{13} + a_{12} \times b_{23} + a_{13} \times b_{33}$$

Examples:

$$A = \begin{pmatrix} 1 & 2 \\ 5 & 6 \end{pmatrix}$$

$$B = \begin{pmatrix} 3 & 4 \\ 7 & 8 \end{pmatrix}$$

$$AB = \begin{pmatrix} (1 \times 3 + 2 \times 7) & (1 \times 4 + 2 \times 8) \\ (5 \times 3 + 6 \times 7) & (5 \times 4 + 6 \times 8) \end{pmatrix} = \begin{pmatrix} 17 & 20 \\ 57 & 68 \end{pmatrix}$$

$$BA = \begin{pmatrix} (3 \times 1 + 4 \times 5) & (3 \times 2 + 4 \times 6) \\ (7 \times 1 + 8 \times 5) & (7 \times 2 + 8 \times 6) \end{pmatrix} = \begin{pmatrix} 23 & 30 \\ 47 & 62 \end{pmatrix}$$

Examples:

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 6 & 7 & 8 \end{pmatrix}$$

$$B = \begin{pmatrix} 4 & 1 \\ 3 & 2 \\ 5 & 3 \end{pmatrix}$$

$$AB = \begin{pmatrix} (1 \times 4 + 2 \times 3 + 3 \times 5) & (1 \times 1 + 2 \times 2 + 3 \times 3) \\ (6 \times 4 + 7 \times 3 + 8 \times 5) & (6 \times 1 + 7 \times 2 + 8 \times 3) \end{pmatrix} = \begin{pmatrix} 25 & 14 \\ 85 & 44 \end{pmatrix}$$

$$BA = \begin{pmatrix} (4 \times 1 + 1 \times 6) & (4 \times 2 + 1 \times 7) & (4 \times 3 + 1 \times 8) \\ (3 \times 1 + 2 \times 6) & (3 \times 2 + 2 \times 7) & (3 \times 3 + 2 \times 8) \\ (5 \times 1 + 3 \times 6) & (5 \times 2 + 3 \times 7) & (5 \times 3 + 3 \times 8) \end{pmatrix} = \begin{pmatrix} 10 & 16 & 20 \\ 15 & 20 & 25 \\ 23 & 31 & 39 \end{pmatrix}$$

Examples:

$$A = (1 \quad 2 \quad 3)$$

$$B = \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix}$$

$$AB = ((1 \times 4 + 2 \times 5 + 3 \times 6)) = (32)$$

$$BA = \begin{pmatrix} (4 \times 1) & (4 \times 2) & (4 \times 3) \\ (5 \times 1) & (5 \times 2) & (5 \times 3) \\ (6 \times 1) & (6 \times 2) & (6 \times 3) \end{pmatrix} = \begin{pmatrix} 4 & 8 & 12 \\ 5 & 10 & 15 \\ 6 & 12 & 18 \end{pmatrix}$$

$$A(AB) = ((1 \times 32) \quad (2 \times 32) \quad (3 \times 32)) = (32 \quad 64 \quad 96)$$

$(AA)B = \text{Illegal! } A \text{ is not square}$

$$\begin{aligned} A(BA) &= ((1 \times 4 + 2 \times 5 + 3 \times 6) \quad (1 \times 8 + 2 \times 10 + 3 \times 15) \quad (1 \times 6 + 2 \times 12 + 3 \times 18)) \\ &= (32 \quad 64 \quad 96) \end{aligned}$$

$$(AB)A = (32 \times 1 \quad 32 \times 2 \quad 32 \times 3) = (32 \quad 64 \quad 96)$$

$B(AA) = \text{Illegal! } A \text{ is not square}$

$(BA)A = \text{Illegal! } BA \text{ has 3 columns but } A \text{ has only 1 row}$

$$\begin{aligned} B(AB) &= \begin{pmatrix} 4 \times 32 \\ 5 \times 32 \\ 6 \times 32 \end{pmatrix} \\ &= \begin{pmatrix} 128 \\ 160 \\ 192 \end{pmatrix} \end{aligned}$$

$$(BA)B = \begin{pmatrix} (4 \times 4 + 8 \times 5 + 12 \times 6) \\ (5 \times 5 + 10 \times 5 + 15 \times 6) \\ (6 \times 4 + 12 \times 5 + 18 \times 6) \end{pmatrix} = \begin{pmatrix} 128 \\ 160 \\ 192 \end{pmatrix}$$

$A(BB) = \text{Illegal! } B \text{ is not square}$

$$(AB)B = \begin{pmatrix} 32 \times 4 \\ 32 \times 5 \\ 32 \times 6 \end{pmatrix} = \begin{pmatrix} 128 \\ 160 \\ 192 \end{pmatrix} \quad 8$$

## Rules of matrix multiplication

- The matrix product  $AB$  is legal iff  $A$  has the same number of columns as  $B$  has rows, i.e.  $A$  is  $(m \times p)$  and  $B$  is  $(p \times n)$ . The resulting matrix is  $(m \times n)$ .
- If  $AB$  is legal,  $BA$  is legal iff  $m = n$ . If so,  $BA$  will be of size  $(p \times p)$ .
- Matrix multiplication is **not commutative**:

$$AB \neq BA$$

$AB$  :  $A$  pre-multiplies  $B$ ;  $B$  post-multiplies  $A$

$BA$  :  $B$  pre-multiplies  $A$ ;  $A$  post-multiplies  $B$

- Matrix multiplication is **associative**:

$$A(BC) = (AB)C$$

- If  $A$  is  $(m \times p)$ ,  $B$  is  $(p \times q)$  and  $C$  is  $(q \times n)$ , the product  $ABC$  is  $(m \times n)$ .
- Matrix multiplication is **distributive**:

$$A(B + C) = AB + AC$$

- If  $A$  is  $(m \times p)$  and  $B$  and  $C$  are both  $(p \times n)$ , the result  $A(B + C)$  is  $(m \times n)$ .
- Multiplication by the Identity matrix leaves a matrix unchanged. If  $A$  is  $(m \times n)$ , then

$$I_m A = A$$

$$A I_n = A$$

## Transpose of the matrix product

The transpose of a matrix product is equal to the product of the matrix transposes *with the order of multiplication reversed*. If  $A$  is  $(m \times p)$  and  $B$  is  $(p \times n)$

$$[AB]^T = B^T A^T$$

Note that the matrix  $[AB]^T$  will be  $(n \times m)$ .

## Determinant, minor, cofactor and adjoint

The *determinant* of a  $2 \times 2$  matrix is defined as the difference between the product of the leading diagonal and the other diagonal, i.e. given

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

the determinant of  $A$ , written  $\det(A)$  or  $|A|$  is

$$|A| = \det(A) = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11} \times a_{22} - a_{21} \times a_{12}$$

Given a matrix  $A$ , the *minor*  $M_{ij}$  is the determinant of matrix  $A$  with the  $i^{\text{th}}$  row and  $j^{\text{th}}$  column removed:

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

$$\begin{array}{lll} M_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} & M_{12} = \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} & M_{13} = \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} \\ M_{21} = \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} & M_{22} = \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} & M_{23} = \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix} \\ M_{31} = \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix} & M_{32} = \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix} & M_{33} = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \end{array}$$

The *co-factor* matrix  $C$  is defined as the matrix in which every element  $c_{ij}$  is equal to the minor  $M_{ij}$  times a place sign

$$Sgn = \begin{pmatrix} + & - & + & - & \cdots \\ - & + & - & + & \cdots \\ + & - & + & - & \cdots \\ - & + & - & + & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

i.e.,

$$c_{ij} = (-1)^{i+j} \times M_{ij} \quad \forall i \in [1, m], j \in [1, n]$$

$$C = \begin{pmatrix} + \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} & - \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} & + \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} \\ - \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} & + \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} & - \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix} \\ + \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix} & - \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix} & + \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \end{pmatrix}$$

For a matrix of size  $m \times m$  where  $m$  is greater than 2, the determinant can be found by multiplying the elements of any row or column by the corresponding row or column in the cofactor matrix.

Example:

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 6 & 5 & 4 \\ 7 & 9 & 8 \end{pmatrix}$$

Calculate the co-factor matrix  $C$  and then find the product  $A \times C^T$

$$C = \begin{pmatrix} +4 & -20 & +19 \\ +11 & -13 & +5 \\ -7 & +14 & -7 \end{pmatrix} \quad A^j = C^T = \begin{pmatrix} +4 & +11 & -7 \\ -20 & -13 & +14 \\ +19 & +5 & -7 \end{pmatrix}$$

The transpose of the co-factor matrix  $A^j = C^T$  is called the *adjoint*

$$A \times A^j = A \times C^T = \begin{pmatrix} 1 & 2 & 3 \\ 6 & 5 & 4 \\ 7 & 9 & 8 \end{pmatrix} \begin{pmatrix} +4 & +11 & -7 \\ -20 & -13 & +14 \\ +19 & +5 & -7 \end{pmatrix} = \begin{pmatrix} 21 & 0 & 0 \\ 0 & 21 & 0 \\ 0 & 0 & 21 \end{pmatrix} = 21 \times I_3$$

The value that appears everywhere on the leading diagonal (21 in this example) is the determinant of the matrix

$$|A| = \det(A) = 21$$

To find the determinant of a large matrix ( $m \times m$ ,  $m > 2$ ), it is not necessary in practice to form the full adjoint matrix. The determinant can be found by selecting any row (or column) of the matrix, and summing the product of each element in that row (or column) with its cofactor.

$$\begin{aligned} |A| &= a_{11} \times c_{11} + a_{12} \times c_{12} + a_{13} \times c_{13} + \dots \\ &= a_{21} \times c_{21} + a_{22} \times c_{22} + a_{23} \times c_{23} + \dots \\ &= a_{31} \times c_{31} + a_{32} \times c_{32} + a_{33} \times c_{33} + \dots \\ &= a_{11} \times c_{11} + a_{21} \times c_{21} + a_{31} \times c_{31} + \dots \\ &= a_{12} \times c_{12} + a_{22} \times c_{22} + a_{32} \times c_{32} + \dots \\ &= a_{13} \times c_{13} + a_{23} \times c_{23} + a_{33} \times c_{33} + \dots \\ &= +a_{11} \times M_{11} - a_{12} \times M_{12} + a_{13} \times M_{13} + \dots \\ &= -a_{21} \times M_{21} + a_{22} \times M_{22} - a_{23} \times M_{23} + \dots \\ &= +a_{31} \times M_{31} - a_{32} \times M_{32} + a_{33} \times M_{33} + \dots \\ &= +a_{11} \times M_{11} - a_{21} \times M_{21} + a_{31} \times M_{31} + \dots \\ &= -a_{12} \times M_{12} + a_{22} \times M_{22} - a_{32} \times M_{32} + \dots \\ &= +a_{13} \times M_{13} - a_{23} \times M_{23} + a_{33} \times M_{33} + \dots \end{aligned}$$

## Properties of determinants

- If two rows or columns of matrix  $A$  are equal,  $|A| = 0$

$$\begin{aligned} |A| &= \begin{vmatrix} a & b \\ a & b \end{vmatrix} \\ &= ab - ba \\ &= 0 \end{aligned}$$

- Multiplying a row (or column) by a scalar multiplies the determinant by the same factor

$$\begin{aligned} |A| &= \begin{vmatrix} a & b \\ c & d \end{vmatrix} \\ &= ad - bc \end{aligned}$$

$$\begin{aligned} |B| &= \begin{vmatrix} \lambda a & \lambda b \\ c & d \end{vmatrix} \\ &= \lambda ad - \lambda bc \\ &= \lambda |A| \end{aligned}$$

- Swapping two rows (or columns) changes the sign

$$\begin{aligned} |A| &= \begin{vmatrix} a & b \\ c & d \end{vmatrix} \\ &= ad - bc \end{aligned}$$

$$\begin{aligned} |B| &= \begin{vmatrix} b & a \\ d & c \end{vmatrix} \\ &= bc - ad \\ &= -|A| \end{aligned}$$

- Adding elements to one row (or column) is equivalent to adding two determinants

$$\begin{aligned} |A| &= \begin{vmatrix} a+x & b+y \\ c & d \end{vmatrix} \\ &= (a+x)d - (b+y)c \\ &= (ad - bc) + (xd - yc) \\ &= \begin{vmatrix} a & b \\ c & d \end{vmatrix} + \begin{vmatrix} x & y \\ c & d \end{vmatrix} \end{aligned}$$

- Adding multiples of one row (or column) to another does not change the determinant

$$\begin{aligned} |A| &= \begin{vmatrix} a & b \\ c & d \end{vmatrix} \\ &= ad - bc \end{aligned}$$

$$\begin{aligned} |B| &= \begin{vmatrix} a + \lambda c & b + \lambda d \\ c & d \end{vmatrix} \\ &= \begin{vmatrix} a & b \\ c & d \end{vmatrix} + \begin{vmatrix} \lambda c & \lambda d \\ c & d \end{vmatrix} \\ &= \begin{vmatrix} a & b \\ c & d \end{vmatrix} + \lambda \begin{vmatrix} c & d \\ c & d \end{vmatrix} \\ &= |A| + 0 \end{aligned}$$

- Transposing a matrix does not change the determinant

$$\begin{aligned} |A| &= \begin{vmatrix} a & b \\ c & d \end{vmatrix} \\ &= ad - bc \end{aligned}$$

$$\begin{aligned} |A^T| &= \begin{vmatrix} a & c \\ b & d \end{vmatrix} \\ &= ad - cb \\ &= |A| \end{aligned}$$

- The determinant of a product of two matrices is the same as the product of their determinants

$$\begin{aligned} |A| &= \begin{vmatrix} a & b \\ c & d \end{vmatrix} \\ &= ad - bc \end{aligned}$$

$$\begin{aligned} |B| &= \begin{vmatrix} e & f \\ g & h \end{vmatrix} \\ &= eh - fg \end{aligned}$$

$$\begin{aligned} |AB| &= \begin{vmatrix} ae + bg & af + bh \\ ce + dg & cf + dh \end{vmatrix} \\ &= adeh + bcfg - adfg - bceh \\ &= (ad - bc)(eh - fg) \\ &= |A||B| \end{aligned}$$

## Matrix inverse

Recall that the product of a matrix by its adjoint is equal to the product of its determinant and the Identity matrix

$$A \times A^j = |A| \times I$$

The determinant is a scalar and both sides of the equation may be divided by it

$$A \times \frac{A^j}{|A|} = I$$

The matrix  $A^j/|A|$  is called the *matrix inverse* of A, and is denoted by the symbol  $A^{-1}$

$$A^{-1} = \frac{A^j}{|A|} = \frac{C^T}{|A|}$$

and has the property (if  $|A| \neq 0$ ) that

$$AA^{-1} = I = A^{-1}A$$

## Solution of linear matrix equations

Consider the following matrix equation which has the form  $A\theta = B$

$$\underbrace{\begin{pmatrix} 1 & 2 & 3 \\ 6 & 5 & 4 \\ 7 & 9 & 8 \end{pmatrix}}_A \underbrace{\begin{pmatrix} x \\ y \\ z \end{pmatrix}}_\theta = \underbrace{\begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix}}_B$$

The matrix inverse  $A^{-1}$  can be calculated from previous results

$$A^{-1} = \frac{A^j}{|A|} = \frac{1}{21} \begin{pmatrix} +4 & +11 & -7 \\ -20 & -13 & +14 \\ +19 & +5 & -7 \end{pmatrix} = \begin{pmatrix} +0.19048 & +0.52381 & -0.33333 \\ -0.95238 & -0.61905 & +0.66667 \\ +0.90476 & +0.23810 & -0.33333 \end{pmatrix}$$

Pre-multiplying both sides of the equation by the matrix inverse  $A^{-1}$  yields a direct solution to the simultaneous equations

$$\begin{aligned} A \times \theta &= B \\ \underbrace{A^{-1} \times A}_I \times \theta &= A^{-1} \times B \\ I \times \theta &= A^{-1} \times B \\ \begin{pmatrix} x \\ y \\ z \end{pmatrix} &= \theta = A^{-1}B = \begin{pmatrix} +0.19048 & +0.52381 & -0.33333 \\ -0.95238 & -0.61905 & +0.66667 \\ +0.90476 & +0.23810 & -0.33333 \end{pmatrix} \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix} = \begin{pmatrix} +1.09524 \\ -1.47619 \\ +0.95238 \end{pmatrix} \end{aligned}$$

## Gaussian elimination

Although matrix inversion yields a direct answer to linear matrix equations, it can be computationally intensive. An alternative method of solving equations is to use *Gaussian elimination* with *back substitution*.

Consider the process of solving a set of simultaneous linear equations

$$4x + 2y - z = 5 \quad (\text{g1})$$

$$2x + 2y + z = 9 \quad (\text{g2})$$

$$x - y + z = 2 \quad (\text{g3})$$

To solve  $x$ ,  $y$  and  $z$ , we add or subtract multiples of one equation from another to eliminate variables, e.g.

$$4x + 2y - z = 5 \quad (\text{g1})$$

$$4x + 4y + 2z = 18 \quad (\text{g4} = 2 \times \text{g2})$$

$$4x - 4y + 4z = 8 \quad (\text{g5} = 4 \times \text{g3})$$

$$4x + 2y - z = 5 \quad (\text{g1})$$

$$0x + 2y + 3z = 13 \quad (\text{g6} = \text{g4} - \text{g1})$$

$$0x + 6y - 5z = -3 \quad (\text{g7} = \text{g5} - \text{g1})$$

$$4x + 2y - z = 5 \quad (\text{g1})$$

$$0x + 2y + 3z = 13 \quad (\text{g6})$$

$$0x + 2y - \frac{5}{3}z = -1 \quad (\text{g8} = \frac{1}{3} \times \text{g7})$$

$$4x + 2y - z = 5 \quad (\text{g1})$$

$$0x + 2y + 3z = 13 \quad (\text{g6})$$

$$0x + 0y + \frac{14}{3}z = 14 \quad (\text{g9} = \text{g6} - \text{g8})$$

$$1x + \frac{1}{2}y - \frac{1}{4}z = \frac{5}{4} \quad (\text{g10} = \frac{1}{4} \times \text{g1})$$

$$0x + 1y + \frac{3}{2}z = \frac{13}{2} \quad (\text{g11} = \frac{1}{2} \times \text{g6})$$

$$0x + 0y + 1z = 3 \quad (\text{g12} = \frac{3}{14} \times \text{g9})$$

The operations can be expressed using *augmented matrix* notation:

$$G = \left( \begin{array}{ccc|c} 4 & 2 & -1 & 5 \\ 2 & 2 & 1 & 9 \\ 1 & -1 & 1 & 2 \end{array} \right)$$

Rows of the matrix may be multiplied by a scalar, added to each other, or subtracted from each other

$$G = \left( \begin{array}{ccc|c} 4 & 2 & -1 & 5 \\ 4 & 4 & 2 & 18 \\ 4 & -4 & 4 & 8 \end{array} \right)$$

$$= \left( \begin{array}{ccc|c} 4 & 2 & -1 & 5 \\ 0 & 2 & 3 & 13 \\ 0 & 6 & 5 & -3 \end{array} \right)$$

$$= \left( \begin{array}{ccc|c} 4 & 2 & -1 & 5 \\ 0 & 2 & 3 & 13 \\ 0 & 0 & \frac{5}{3} & -1 \end{array} \right)$$

$$= \left( \begin{array}{ccc|c} 4 & 2 & -1 & 5 \\ 0 & 2 & 3 & 13 \\ 0 & 0 & \frac{14}{3} & 14 \end{array} \right)$$

$$= \left( \begin{array}{ccc|c} 1 & \frac{1}{2} & -\frac{1}{4} & \frac{5}{4} \\ 0 & 1 & \frac{3}{2} & \frac{13}{2} \\ 0 & 0 & 1 & 3 \end{array} \right)$$

The final form of the matrix is upper triangular, with 1 as the first non-zero element on each row

$$\left( \begin{array}{ccc|c} 1 & \frac{1}{2} & -\frac{1}{4} & \frac{5}{4} \\ 0 & 1 & \frac{3}{2} & \frac{13}{2} \\ 0 & 0 & 1 & 3 \end{array} \right)$$

This form of matrix is known as *row echelon*. Using the row echelon matrix, the unknown variables can be obtained directly using back substitution.

From row 3 (g12)

$$z = 3$$

From row 2 (g11)

$$\begin{aligned} y + \frac{3}{2}z &= \frac{13}{2} \\ y &= \frac{13}{2} - \frac{3}{2}z \\ &= \frac{13}{2} - \frac{3}{2} \times 3 \\ &= \frac{13 - 9}{2} \\ &= 2 \end{aligned}$$

From row 1 (g10)

$$\begin{aligned} x + \frac{1}{2}y - \frac{1}{4}z &= \frac{5}{4} \\ x &= \frac{5}{4} - \frac{1}{2}y + \frac{1}{4}z \\ &= \frac{5}{4} - \frac{1}{2} \times 2 + \frac{1}{4} \times 3 \\ &= \frac{5 - 4 + 3}{4} \\ &= 1 \end{aligned}$$

## Gauss-Jordan elimination

Gaussian elimination can also be used to find a matrix inverse, by augmenting the matrix with the Identity matrix and reducing the original matrix down to  $I$

$$\begin{aligned}
 (A|I) &= \left( \begin{array}{cc|cc} 4 & 3 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{array} \right) \\
 &= \left( \begin{array}{cc|cc} 4 & 3 & 1 & 0 \\ 4 & 8 & 0 & 4 \end{array} \right) && \text{(row 2} \leftarrow \text{row 2} \times 4) \\
 &= \left( \begin{array}{cc|cc} 4 & 3 & 1 & 0 \\ 0 & 5 & -1 & 4 \end{array} \right) && \text{(row 2} \leftarrow \text{row 2} - \text{row 1)} \\
 &= \left( \begin{array}{cc|cc} 4 & 3 & 1 & 0 \\ 0 & 1 & \frac{-1}{5} & \frac{4}{5} \end{array} \right) && \text{(row 2} \leftarrow \text{row 2} \div 5) \\
 &= \left( \begin{array}{cc|cc} \frac{4}{3} & 1 & \frac{1}{3} & 0 \\ 0 & 1 & \frac{-1}{5} & \frac{4}{5} \end{array} \right) && \text{(row 1} \leftarrow \text{row 1} \div 3) \\
 &= \left( \begin{array}{cc|cc} \frac{4}{3} & 0 & \frac{8}{15} & \frac{-4}{5} \\ 0 & 1 & \frac{-1}{5} & \frac{4}{5} \end{array} \right) && \text{(row 1} \leftarrow \text{row 1} - \text{row 2)} \\
 &= \left( \begin{array}{cc|cc} 1 & 0 & \frac{2}{5} & \frac{-3}{5} \\ 0 & 1 & \frac{-1}{5} & \frac{4}{5} \end{array} \right) && \text{(row 1} \leftarrow \text{row 1} \div \frac{4}{3}) \\
 &= ( I \quad | \quad A^{-1} )
 \end{aligned}$$

## Transformation matrices: rotation

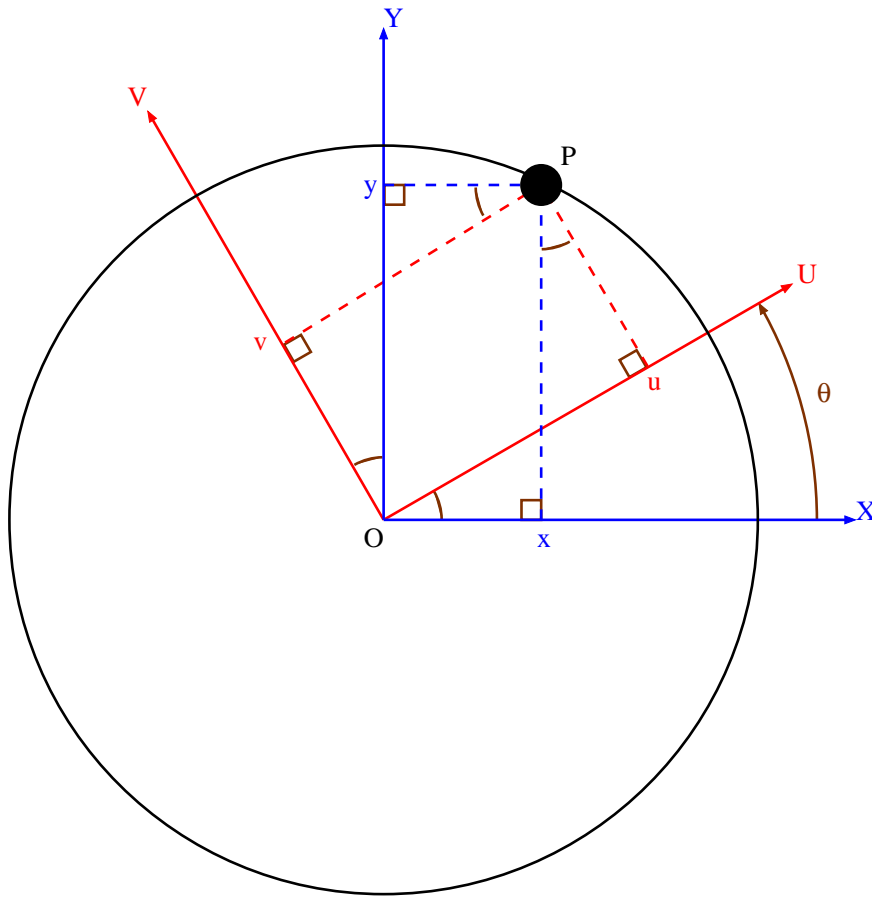


Figure 1: A point P is rotated in a Cartesian axis set

Consider a disk centred at point O on a Cartesian axis set  $(X, Y)$ . A point P on the disk is located at position  $(x, y)$  as shown in Figure 1. If the disk is rotated by  $\theta$  radians clockwise, point P will move relative to the axis set  $(X, Y)$ . An equivalent effect can be seen if the point P is considered to be fixed and the axis set is rotated anti-clockwise, to become axis set  $(U, V)$ . Given the initial co-ordinates of P  $(x, y)$  and the angle of rotation  $\theta$ , we wish to find the new co-ordinates  $(u, v)$ . Figure 2 shows the salient geometric aspects of the problem. The following relationships can be obtained by inspection:

$$\begin{aligned} x &= \overrightarrow{OQ} & y &= \overrightarrow{PR} + \overrightarrow{RQ} \\ u &= \overrightarrow{OR} + \overrightarrow{RS} & v &= \overrightarrow{PS} \end{aligned}$$

Furthermore, by trigonometry,

$$\cos \theta = \frac{\overrightarrow{OQ}}{\overrightarrow{OR}} = \frac{\overrightarrow{PS}}{\overrightarrow{PR}} \quad \sin \theta = \frac{\overrightarrow{RQ}}{\overrightarrow{OR}} = \frac{\overrightarrow{RS}}{\overrightarrow{PR}} \quad \tan \theta = \frac{\overrightarrow{RQ}}{\overrightarrow{OQ}} = \frac{\overrightarrow{RS}}{\overrightarrow{PS}}$$

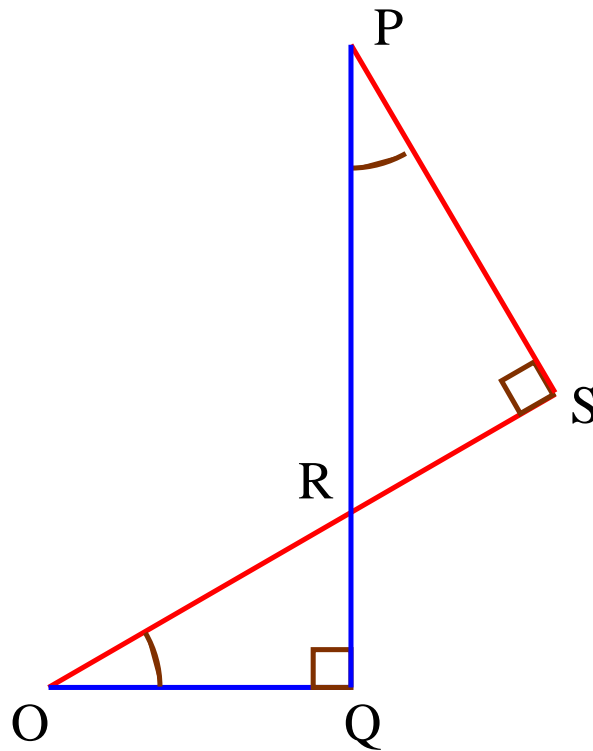


Figure 2: Detail showing lines and angles of interest

Using these relationships, the required expressions can be derived:

$$\begin{aligned}
 u &= \vec{OR} + \vec{RS} \\
 &= \vec{OR} + \vec{PR} \sin \theta && \text{using } \sin \theta = \frac{\vec{RS}}{\vec{PR}} \\
 &= \vec{OR} + (y - \vec{RQ}) \sin \theta && \text{using } y = \vec{PR} + \vec{RQ} \\
 &= \vec{OR} + (y - \vec{OR} \sin \theta) \sin \theta && \text{using } \sin \theta = \frac{\vec{RQ}}{\vec{OR}} \\
 &= \vec{OR}(1 - \sin^2 \theta) + y \sin \theta && \text{collecting terms} \\
 &= \vec{OR} \cos^2 \theta + y \sin \theta && \text{using } \sin^2 \theta + \cos^2 \theta \equiv 1 \\
 &= \frac{\vec{OQ}}{\cos \theta} \cos^2 \theta + y \sin \theta && \text{using } \cos \theta = \frac{\vec{OQ}}{\vec{OR}} \\
 &= \vec{OQ} \cos \theta + y \sin \theta && \text{cancelling terms of } \cos \theta \\
 &= x \cos \theta + y \sin \theta \\
 v &= \vec{PS} \\
 &= \vec{PR} \cos \theta && \text{using } \cos \theta = \frac{\vec{PS}}{\vec{PR}} \\
 &= (y - \vec{RQ}) \cos \theta && \text{using } y = \vec{PR} + \vec{RQ} \\
 &= (y - \vec{OQ} \tan \theta) \cos \theta && \text{using } \tan \theta = \frac{\vec{RQ}}{\vec{OQ}} \\
 &= y \cos \theta - \vec{OQ} \tan \theta \cos \theta && \text{collecting terms} \\
 &= y \cos \theta - \vec{OQ} \sin \theta && \text{cancelling terms of } \cos \theta \\
 &= y \cos \theta - x \sin \theta && \text{using } x = \vec{OQ} \\
 &= -x \sin \theta + y \cos \theta && \text{rearranging terms}
 \end{aligned}$$

Expressed in matrix form, this is

$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

The matrix  $\begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix}$  is called a *Transformation* matrix.

A transformation matrix for rotating the  $(U, V)$  axis set to  $(X, Y)$  can be derived in a similar manner, but it is simpler to note that this is equivalent to a rotation in the other direction, i.e. with negative  $\theta$ . Noting that  $\sin -\theta = -\sin \theta$  and  $\cos -\theta = \cos \theta$ , the inverse transformation is simply

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}$$

If we consider the rotation in 3 dimensions, the axes must be expanded to form orthogonal right hand sets,  $(X, Y, Z)$  and  $(U, V, W)$  and the point P will be represented by three co-ordinates relative to these,  $(x, y, z)$  and  $(u, v, w)$  respectively. For planar rotation of the disk about the  $Z$  axis,  $z = w$ , hence the transformation matrices are

$$\begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} \cos \psi & \sin \psi & 0 \\ -\sin \psi & \cos \psi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} \quad \text{for clockwise rotation } \psi \text{ of the disk about the } Z \text{ axis}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \cos \psi & -\sin \psi & 0 \\ \sin \psi & \cos \psi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} u \\ v \\ w \end{pmatrix} \quad \text{for counter-clockwise rotation } \psi \text{ of the disk about the } Z \text{ axis}$$

Similar results can be derived for rotation about the  $X$  axis

$$\begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos \phi & \sin \phi \\ 0 & -\sin \phi & \cos \phi \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} \quad \text{for clockwise rotation } \phi \text{ of the disk about the } X \text{ axis}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos \phi & -\sin \phi \\ 0 & \sin \phi & \cos \phi \end{pmatrix} \begin{pmatrix} u \\ v \\ w \end{pmatrix} \quad \text{for counter-clockwise rotation } \phi \text{ of the disk about the } X \text{ axis}$$

and similarly for rotation about the  $Y$  axis

$$\begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} \cos \theta & 0 & -\sin \theta \\ 0 & 1 & 0 \\ \sin \theta & 0 & \cos \theta \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} \quad \text{for clockwise rotation } \theta \text{ of the disk about the } Y \text{ axis}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \cos \theta & 0 & \sin \theta \\ 0 & 1 & 0 \\ -\sin \theta & 0 & \cos \theta \end{pmatrix} \begin{pmatrix} u \\ v \\ w \end{pmatrix} \quad \text{for counter-clockwise rotation } \theta \text{ of the disk about the } Y \text{ axis}$$

## Multiple rotations

Define three transformation matrices,  $T_x$ ,  $T_y$  and  $T_z$ , which rotate a point  $\frac{\pi}{4}$  radians clockwise about the  $X$ ,  $Y$  and  $Z$  axes respectively.

$$T_x = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \quad T_y = \begin{pmatrix} \frac{-1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{pmatrix} \quad T_z = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Consider the effect of applying multiple transformations on a point  $P_0$  located initially on the  $X$  axis at  $(1, 0, 0)$ .

If  $P_0$  is rotated first about the  $Z$  axis, using  $T_z$ , then the new co-ordinates will be

$$P_1 = T_z P_0 = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} \\ 0 \end{pmatrix}$$

Then rotate  $P_1$  about the  $X$  axis using  $T_x$

$$P_2 = T_x P_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{pmatrix}$$

Note that the same effect could be achieved by multiplying the position  $P$  by the product of both transformation matrices

$$P_2 = T_x T_z P_0$$

Now consider the effect on the original position  $P_0$  if the order of application is reversed, i.e. rotation about  $X$  occurs before rotation about  $Z$ ,

$$P_3 = T_x P_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$P_4 = T_z P_3 = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} \\ 0 \end{pmatrix}$$

This second scenario is equivalent to multiplying the position  $P_0$  by the matrix product  $T_z T_x$ , which is the reverse of the first scenario.

$$P_4 = T_z T_x P_0$$

NB:  $P_4 \neq P_2$  !!!

In general, a sequence of transformations can be applied, but the order of application is significant. The right-most transformation matrix is the *first* transformation applied; the left-most matrix in the product is the *last* transformation applied.

$$P' = \underbrace{T_3 T_2}_{\text{Last}} \underbrace{T_1}_{\text{First}} P$$