December questions

Q1(a) A point P lies in an orthogonal right-handed axis $\{x, y, z\}$. Explain the effect on P of each of the three transformation matrices:

$$\tau_1(\alpha) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha \\ 0 & \sin\alpha & \cos\alpha \end{pmatrix} \qquad \tau_2(\beta) = \begin{pmatrix} \cos\beta & 0 & \sin\beta \\ 0 & 1 & 0 \\ -\sin\beta & 0 & \cos\beta \end{pmatrix} \qquad \tau_3(\gamma) = \begin{pmatrix} \cos\gamma & -\sin\gamma & 0 \\ \sin\gamma & \cos\gamma & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

[3 marks]

Q1(b) Find an expression τ to perform the following transformations, and the position of the transformed point $P' = \tau P$, given that $P = (1, 1, 1)^T$.

(i) A rotation of $\pi/2$ radians about the x axis followed by a rotation of $\pi/2$ radians about the z axis.

[4 marks]

(ii) A rotation of $\pi/2$ radians about the z axis followed by a rotation of $\pi/2$ radians about the x axis.

[3 marks]

A1(a) The transformation matrices τ_1, τ_2 and τ_3 rotate the point α, β and γ radians about the x, y and z axes respectively.

A1(b)(i)

$$\tau = \tau_3(\pi/2) \times \tau_1(\pi/2) \qquad \text{No further simplification required for full marks}$$
$$= \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$$
$$= \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$
$$P' = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

A1(b)(ii)

$$\tau = \tau_1(\pi/2) \times \tau_3(\pi/2) \qquad \text{No further simplification required for full marks}$$
$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$
$$= \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 1 & 0 & 0 \end{pmatrix}$$
$$P' = \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}$$

Q2(a) Given the matrix $A = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 0 & 1 \\ 1 & 2 & 4 \end{pmatrix}$, find

(i) C, the cofactor matrix of A

(ii) A^j, the adjoint matrix of A

(ii)i $|A|$, the determinant of A

(iv) A^{-1}, the inverse of A

5 marks

Q2(b) Solve the linear matrix equation $Ab = P$ to find b given that $P = \big(1, 2, 3\big)^T$.

5 marks

A2(a)(i) Cofactor matrix

$$C = \begin{pmatrix} +\begin{vmatrix} 0 & 2 \\ 1 & 4 \end{vmatrix} & -\begin{vmatrix} 1 & 1 \\ 1 & 4 \end{vmatrix} & +\begin{vmatrix} 1 & 1 \\ 0 & 2 \end{vmatrix} \\ -\begin{vmatrix} 2 & 2 \\ 3 & 4 \end{vmatrix} & +\begin{vmatrix} 1 & 1 \\ 3 & 4 \end{vmatrix} & -\begin{vmatrix} 1 & 1 \\ 2 & 2 \end{vmatrix} \\ +\begin{vmatrix} 2 & 0 \\ 3 & 1 \end{vmatrix} & -\begin{vmatrix} 1 & 1 \\ 3 & 1 \end{vmatrix} & +\begin{vmatrix} 1 & 1 \\ 2 & 0 \end{vmatrix} \end{pmatrix}$$
$$= \begin{pmatrix} -2 & -3 & +2 \\ -2 & +1 & 0 \\ +2 & +2 & -2 \end{pmatrix}$$

A2(a)(ii)

$$Adj(A) = C^T$$
$$= \begin{pmatrix} -2 & -2 & +2 \\ -3 & +1 & +2 \\ +2 & 0 & -2 \end{pmatrix}$$

A2(a)(iii) Expanding along the top row ($A_{1,i} \times C^T_{i,1}$)

$$|A| = \begin{pmatrix} 1 & 2 & 3 \end{pmatrix} \begin{pmatrix} -2 \\ -3 \\ 2 \end{pmatrix}$$
$$= -2$$

A2(a)(iv)

$$A^{-1} = \frac{C^T}{|A|}$$
$$= \begin{pmatrix} +1 & +1 & -1 \\ +1.5 & -0.5 & -1 \\ -1 & 0 & +1 \end{pmatrix}$$

A2(b)

$$b = A^{-1}P$$
$$= \begin{pmatrix} +1 & +1 & -1 \\ +1.5 & -0.5 & -1 \\ -1 & 0 & +1 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$
$$= \begin{pmatrix} 0 \\ -2.5 \\ 2 \end{pmatrix}$$

Resit questions

Q3(a) A point P lies in an orthogonal right-handed axis $\{x, y, z\}$. Explain the effect on P of each of the three transformation matrices:

$$\tau_1(\alpha) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha \\ 0 & \sin\alpha & \cos\alpha \end{pmatrix} \qquad \tau_2(\beta) = \begin{pmatrix} \cos\beta & 0 & \sin\beta \\ 0 & 1 & 0 \\ -\sin\beta & 0 & \cos\beta \end{pmatrix} \qquad \tau_3(\gamma) = \begin{pmatrix} \cos\gamma & -\sin\gamma & 0 \\ \sin\gamma & \cos\gamma & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

[3 marks]

Q3(b) Find an expression τ to perform the following transformations, and the position of the transformed point $P' = \tau P$, given that $P = (1, 1, 1)^T$.

(i) A rotation of $\pi/2$ radians about the x axis followed by a rotation of $\pi/2$ radians about the y axis.

[4 marks]

(ii) A rotation of $\pi/2$ radians about the y axis followed by a rotation of $\pi/2$ radians about the x axis.

[3 marks]

A3(a) The transformation matrices τ_1, τ_2 and τ_3 rotate the point α, β and γ radians about the x, y and z axes respectively.

A3(b)(i)

$$\tau = \tau_2(\pi/2) \times \tau_1(\pi/2) \qquad \text{No further simplification required for full marks}$$

$$= \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ -1 & 0 & 0 \end{pmatrix}$$

$$P' = \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}$$

A3(b)(ii)

$$\tau = \tau_1(\pi/2) \times \tau_2(\pi/2) \qquad \text{No further simplification required for full marks}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

$$P' = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

Q4(a) Given the matrix $A = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 0 & 1 \\ 1 & 2 & 4 \end{pmatrix}$, find

(i) C, the cofactor matrix of A

(ii) A^j, the adjoint matrix of A

(ii)i $|A|$, the determinant of A

(iv) A^{-1}, the inverse of A

5 marks

Q4(b) Solve the linear matrix equation $Ab = P$ to find b given that $P = \left(1, 3, 2\right)^T$.

5 marks

A4(a)(i) Cofactor matrix

$$C = \begin{pmatrix} +\begin{vmatrix} 0 & 1 \\ 2 & 3 \end{vmatrix} & -\begin{vmatrix} 1 & 1 \\ 1 & 3 \end{vmatrix} & +\begin{vmatrix} 1 & 0 \\ 1 & 2 \end{vmatrix} \\ -\begin{vmatrix} 2 & 4 \\ 2 & 3 \end{vmatrix} & +\begin{vmatrix} 1 & 4 \\ 1 & 3 \end{vmatrix} & -\begin{vmatrix} 1 & 2 \\ 1 & 2 \end{vmatrix} \\ +\begin{vmatrix} 2 & 4 \\ 0 & 1 \end{vmatrix} & -\begin{vmatrix} 1 & 4 \\ 1 & 1 \end{vmatrix} & +\begin{vmatrix} 1 & 2 \\ 1 & 0 \end{vmatrix} \end{pmatrix}$$
$$= \begin{pmatrix} -2 & -2 & +2 \\ +2 & -1 & 0 \\ +2 & +3 & -2 \end{pmatrix}$$

A4(a)(ii)

$$Adj(A) = C^T$$
$$= \begin{pmatrix} -2 & +2 & +2 \\ -2 & -1 & +3 \\ +2 & 0 & -2 \end{pmatrix}$$

A4(a)(iii) Expanding along the top row ($A_{1,i} \times C^T_{i,1}$)

$$|A| = \begin{pmatrix} 1 & 2 & 4 \end{pmatrix} \begin{pmatrix} -2 \\ -2 \\ 2 \end{pmatrix}$$
$$= +2$$

A4(a)(iv)

$$A^{-1} = \frac{C^T}{|A|}$$
$$= \begin{pmatrix} -1 & +1 & +1 \\ -1 & -0.5 & +1.5 \\ +1 & 0 & -1 \end{pmatrix}$$

A4(b)

$$b = A^{-1}P$$
$$= \begin{pmatrix} -1 & +1 & +1 \\ -1 & -0.5 & +1.5 \\ +1 & 0 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix}$$
$$= \begin{pmatrix} 4 \\ 0.5 \\ -1 \end{pmatrix}$$