Q1. (a) Use the definition of the Laplace transform to obtain the transform $F(s) = \mathcal{L}\{f(t)\}$ where $f(t)$ is the causal function

$$f(t) = 4 + 3t \qquad\qquad ,t \geq 0$$

State the region of convergence.

[5 Marks]

Answer:

$$\begin{aligned} F(s) &= \int_0^\infty e^{-st} f(t) dt \\ &= \int_0^\infty (4+3t) e^{-st} dt \end{aligned}$$

Integration by parts: $\int u dv = uv - \int v du$

$$\begin{aligned} F(s) &= \left[(4+3t)\frac{e^{-st}}{-s}\right]_0^\infty - \int_0^\infty \frac{e^{-st}}{-s} 3dt \\ &= \left[(4+3t)\frac{e^{-st}}{-s}\right]_0^\infty - \left[3\frac{e^{-st}}{s^2}\right]_0^\infty \\ &= \frac{4}{s} + \frac{3}{s^2} \end{aligned}$$

$Re(s) > 0$

(b) Find the inverse Laplace transform

$$f(t) = \mathcal{L}^{-1}\{F(s)\}$$

where

$$F(s) = \frac{s-2}{(s+1)(s+4)}$$

[5 Marks]

Answer:

$$\begin{aligned} F(s) &= \frac{A}{s+1} + \frac{B}{s+4} \\ &= \frac{-1}{s+1} + \frac{2}{s+4} \qquad \text{(Quick cover-up)} \\ f(t) &= -\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\} + 2\mathcal{L}^{-1}\left\{\frac{1}{s+4}\right\} \\ &= -e^{-t} + 2e^{-4t} \end{aligned}$$

(c) Using the Laplace transform method, solve for $t \geq 0$ the following differential equation:

$$\ddot{y} + 4\dot{y} + 5y = 125t^2$$
$$\text{with } \dot{y} = 0, y = 42 \text{ at time } t = 0$$

[10 Marks]

Answer:

$$\text{Using } X(s) = \frac{U(s) + (as+b)x_0 + av_0}{as^2 + bs + c}$$
$$a = 1, b = 4, c = 5, x_0 = 42, v_0 = 0$$
$$u(t) = 125t^2$$
$$\implies U(s) = \mathcal{L}\{125t^2\} = 125\frac{2!}{s^3}$$
$$\therefore Y(s) = \frac{125\frac{2}{s^3} + (s+4)42}{s^2 + 4s + 5}$$
$$= \frac{42s^4 + 168s^3 + 250}{s^3(s^2 + 4s + 5)}$$

Separating into partial fractions

$$\frac{42s^4 + 168s^3 + 250}{s^3(s^2 + 4s + 5)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s^3} + \frac{Ds + E}{s^2 + 4s + 5}$$
$$\implies 42s^4 + 168s^3 + 150 = (As^2 + Bs + C)(s^2 + 4s + 5) + (Ds + E)s^3$$

Equating coefficients of s:

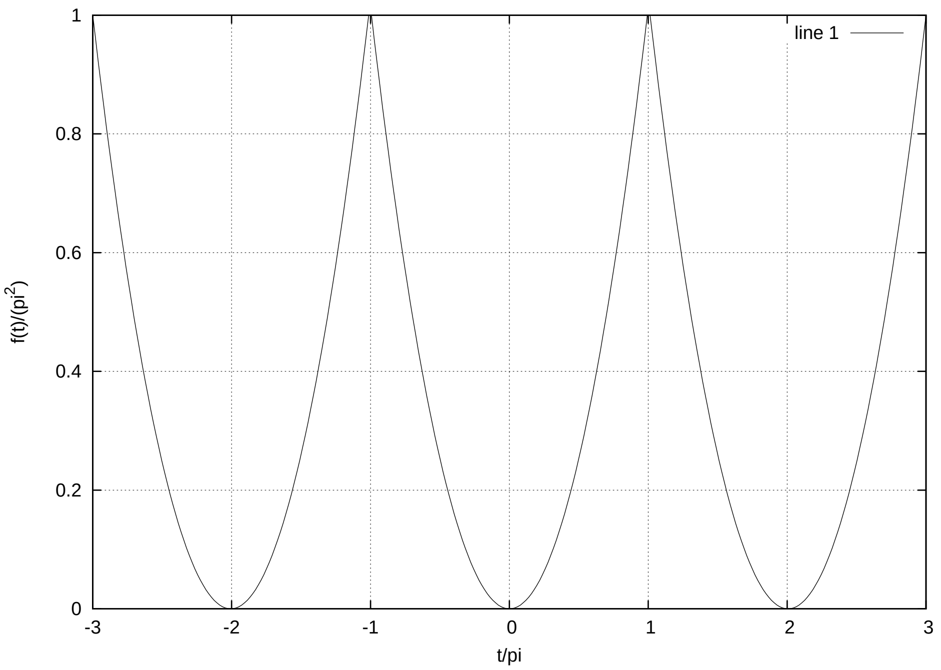
$$s^0: \quad 250 = 5C \quad \implies C = 50$$
$$s^1: \quad 0 = 5B + 4C \quad \implies B = -40$$
$$s^2: \quad 0 = 5A + 4B + C \quad \implies A = 22$$
$$s^3: \quad 168 = 4A + B + E \quad \implies E = 120$$
$$s^4: \quad 42 = A + D \quad \implies D = 20$$

$$Y(s) = \frac{22}{s} - \frac{40}{s^2} + \frac{50}{s^3} + \frac{20s + 120}{s^2 + 4s + 5}$$
$$= \frac{22}{s} - \frac{40}{s^2} + \frac{50}{s^3} + \frac{20(s+2) + 80}{(s+2)^2 + 1}$$
$$= \frac{22}{s} - 40\frac{1!}{s^2} + 25\frac{2!}{s^3} + 20\frac{(s+2)}{(s+2)^2 + 1} + 80\frac{1}{(s+2)^2 + 1}$$
$$y(t) = \mathcal{L}^{-1}\{Y(s)\}$$
$$= 22 - 40t + 25t^2 + e^{-2t}(20\cos t + 80\sin t)$$

Q2. A periodic function $f(t)$ with period 2π is defined by

$$f(t) = t^2 \qquad -\pi < t < \pi, f(t) = f(t + 2\pi)$$

(a) Sketch a graph of $f(t)$ for the interval $-3\pi < t < 3\pi$ and state whether the function is odd or even.



[5 Marks]

Answer: Glyn James pp826-827 The function is even.

(b) Obtain the Fourier transform of $f(t)$.

[15 Marks]

Answer: The function is even $\implies b_n = 0$.

$$f(t) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos nt$$

with

$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(t)dt \qquad = \frac{2}{\pi}\int_0^{\pi} t^2 dt \qquad = \frac{2}{\pi}\left[\frac{t^3}{3}\right]_0^{\pi} \qquad = \frac{2}{3}\pi^2$$

and

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(t)\cos nt dt \qquad (n = 1, 2, 3, ...) \quad = \frac{2}{\pi}\int_0^{\pi} t^2 \cos nt dt$$
$$= \frac{2}{\pi}\left[\frac{t^2}{n}\sin nt + \frac{2t}{n^2}\cos nt - \frac{2}{n^3}\sin nt\right]_0^{\pi} \quad = \frac{2}{\pi}\left(\frac{2\pi}{n^2}\cos n\pi\right) = \frac{4}{n^2}(-1)^n$$

since $\sin n\pi = 0$ and $\cos n\pi = (-1)^n$

$$f(t) = \frac{1}{3}\pi^2 + 4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos nt \quad = \frac{1}{3}\pi^2 - 4\cos t + \cos 2t - \frac{4}{9}\cos 3t + ...$$