

Question 1.

(a) By solving the auxiliary equation find the complementary function of

$$\ddot{x} + 3\dot{x} + 2x = f(t)$$

(b) Find the particular integral of the differential equation if

$$f(t) = e^{-3t}$$

(c) Hence find the full general solution given that

$$x(0) = 1 \quad \text{and} \quad \dot{x}(0) = 0$$

(d) Briefly describe the behaviour of the system as $t \rightarrow \infty$.

[Each part 5 marks]

Solution 1.

(a) Auxiliary equation

$$\lambda^2 + 3\lambda + 2 = 0$$

$$\lambda = \frac{-3 \pm \sqrt{3^2 - 4 \times 1 \times 2}}{2 \times 1} = \frac{-3 \pm \sqrt{1}}{2} = -1.5 \pm 0.5$$

Complementary function

$$x_{cf}(t) = Ae^{-t} + Be^{-2t}$$

(b) Particular integral

$$x_{pi}(t) = Ce^{-3t} \implies \dot{x}_{pi}(t) = -3Ce^{-3t} \quad \text{and} \quad \ddot{x}_{pi}(t) = 9Ce^{-3t}$$

Substitute into the differential equation

$$(9 - 3 \times 3 + 2 \times 1)Ce^{-3t} = e^{-3t}$$

Equate coefficients of e^{-3t}

$$e^{-3t} : \quad (9 - 9 + 2)C = 1 \quad \implies C = 1/2$$

$$x_{pi}(t) = 0.5e^{-3t}$$

(c) General solution

$$\begin{aligned} x(t) &= x_{cf}(t) + x_{pi}(t) \\ &= Ae^{-t} + Be^{-2t} + 0.5e^{-3t} \end{aligned}$$

$$x(0) = 1 = A + B + 0.5$$

$$\implies A = 0.5 - B$$

$$\dot{x}(t) = -Ae^{-t} - 2Be^{-2t} - 1.5e^{-3t}$$

$$\dot{x}(0) = 0 = -A - 2B - 1.5$$

$$= 0 = -(0.5 - B) - 2B - 1.5$$

$$\implies B = -2 \implies A = 2.5$$

$$x(t) = 2.5e^{-t} - 2e^{-2t} + 0.5e^{-3t}$$

(d) All three exponential terms decay to zero as t tends to infinity

$$x(t)_{t \rightarrow \infty} \rightarrow 0.$$

Question 2.

- (a) A mass-spring-damper has equation of motion

$$m\ddot{x} + b\dot{x} + kx = 0$$

where m is the mass, b is the damping coefficient and k is the spring constant.

State expressions for the values of b that would make the system

- (i) Under-damped
- (ii) Critically-damped
- (iii) Over-damped

[3 marks]

- (b) Find an expression for the position $x(t)$ at time t if $m = 1$ kg, $b = 2$ N s/m and $k = 10$ N/m.

[5 marks]

- (c) **Sketch** the response of the system if it is released from rest at a displacement $x(0) = 0.3$ m. Highlight the important features on your sketch.

[7 marks]

- (d) Briefly describe the change in behaviour of the system in response to a similar displacement if the spring constant is reduced to 1 N/m.

[5 marks]

Solution 2.

- (i) Under-damped $b < 2\sqrt{mk}$
(a) (ii) Critically-damped $b = 2\sqrt{mk}$
(iii) Over-damped $b > 2\sqrt{mk}$

(b) Auxiliary equation

$$m\lambda^2 + b\lambda + k = \lambda^2 + 2\lambda + 10 = 0$$

$$\lambda = \frac{-2 \pm \sqrt{2^2 - 40}}{2} = -1 \pm j3$$

Complementary function

$$x_{cf}(t) = e^{-t} (A \sin(3t) + B \cos(3t))$$

Particular integral

$$x_{pi}(t) = 0 \implies \dot{x}_{pi}(t) = 0 \text{ and } \ddot{x}_{pi} = 0$$

$$\boxed{x(t) = e^{-t} (A \sin(3t) + B \cos(3t))}$$

- (c) The sketch should show an oscillatory response, starting at $x(0) = 0.3$ with decreasing amplitude as t increases (red ' $k = 10$ ' line in the plot below). Obtaining A and B from the initial conditions, $x(0) = 0.3$ and $\dot{x}(0) = 0$, the exact equation of the curve is

$$x(t) = e^{-t} (0.1 \sin(3t) + 0.3 \cos(3t))$$

but this is not necessary for full marks

- (d) The reduction in spring constant will make the system critically damped, thus stopping it from oscillating (green ' $k = 1$ ' line below). The exact equation (not required) is

$$x(t) = 0.3(1 + t)e^{-t}$$

