Question 1.

(a) By solving the auxiliary equation find the complementary function of

$$\ddot{x} + 7\dot{x} + 10x = f(t)$$

(b) Find the particular integral of the differential equation if

$$f(t) = e^{-7t}$$

(c) Hence find the full general solution given that

$$x(0) = 2.1 \quad \text{and} \quad \dot{x}(0) = -7.7$$

(d) Briefly describe the behaviour of the system as $t \to \infty$.

[Each part 5 marks]

Solution 1.

(a) Auxiliary equation

$$\lambda^2 + 7\lambda + 10 = 0$$

$$\lambda = \frac{-7 \pm \sqrt{7^2 - 4 \times 1 \times 10}}{2 \times 1} \qquad = \frac{-7 \pm \sqrt{9}}{2} \qquad = -3.5 \pm 1.5$$

Complementary function

$\boxed{x_{cf}(t) = Ae^{-2t} + Be^{-5t}}$

(b) Particular integral

$$x_{pi}(t) = Ce^{-7t} \qquad \Longrightarrow \dot{x}_{pi}(t) = -7Ce^{-7t} \qquad \text{and} \qquad \ddot{x}_{pi}(t) = 49Ce^{-3t}$$

Substitute into the differential equation

$$(49 - 7 \times 7 + 10 \times 1)Ce^{-7t} = e^{-7t}$$

Equate coefficients of e^{-7t}

$$e^{-3t}: \qquad (49 - 49 + 10)C = 1 \qquad \Longrightarrow C = 1/10$$

$\boxed{x_{pi}(t) = 0.1e^{-7t}}$

(c) General solution

$$\begin{aligned} x(t) &= x_{cf}(t) + x_{pi}(t) \\ &= Ae^{-2t} + Be^{-5t} + 0.1e^{-7t} \end{aligned}$$

$$\begin{aligned} &x(0) = 2.1 = A + B + 0.1 \\ &\Longrightarrow A = 2 - B \end{aligned}$$

$$\dot{x}(t) = -2Ae^{-2t} - 5Be^{-5t} - 0.7e^{-7t}$$

$$\begin{aligned} \dot{x}(0) &= -7.7 = -2A - 5B - 0.7 \\ &= -7.7 = -2 * (2 - B) - 5B - 0.7 \\ &\Longrightarrow \; B = 1 \; \Longrightarrow \; A = 1 \end{aligned}$$

$\boxed{x(t) = e^{-2t} + e^{-5t} + 0.1e^{-7t}}$

(d) All three exponential terms decay to zero as t tends to infinity

$\boxed{x(t)_{t \to \infty} \to 0}$.

Question 2.

(a) A mass-spring-damper has equation of motion

$$m\ddot{x} + b\dot{x} + kx = 0$$

where m is the mass, b is the damping coefficient and k is the spring constant.

State expressions for the values of b that would make the system

(i) Under-damped

(ii) Critically-damped

(iii) Over-damped

[3 marks]

(b) Find an expression for the position $x(t)$ at time t if $m = 1$ kg, $b = 2$ N s/m and $k = 2$ N/m.
[5 marks]

(c) **Sketch** the response of the system if it is released from rest at a displacement $x(0) = 0.1$ m. Highlight the important features on your sketch.
[7 marks]

(d) Briefly describe the change in behaviour of the system in response to a similar displacement if the spring constant is halved to 1 N/m.
[5 marks]

Solution 2.

(a)
(i) Under-damped $b < 2\sqrt{mk}$
(ii) Critically-damped $b = 2\sqrt{mk}$
(iii) Over-damped $b > 2\sqrt{mk}$

(b) Auxiliary equation

$$m\lambda^2 + b\lambda + k = \lambda^2 + 2\lambda + 2 = 0$$

$$\lambda = \frac{-2 \pm \sqrt{2^2 - 8}}{2} = -1 \pm j$$

Complementary function

$$x_{cf}(t) = e^{-t}\left(A\sin(t) + B\cos(t)\right)$$

Particular integral

$$x_{pi}(t) = 0 \implies \dot{x}_{pi}(t) = 0 \text{ and } \ddot{x}_{pi} = 0$$

$$\boxed{x(t) = e^{-t}\left(A\sin(t) + B\cos(t)\right)}$$

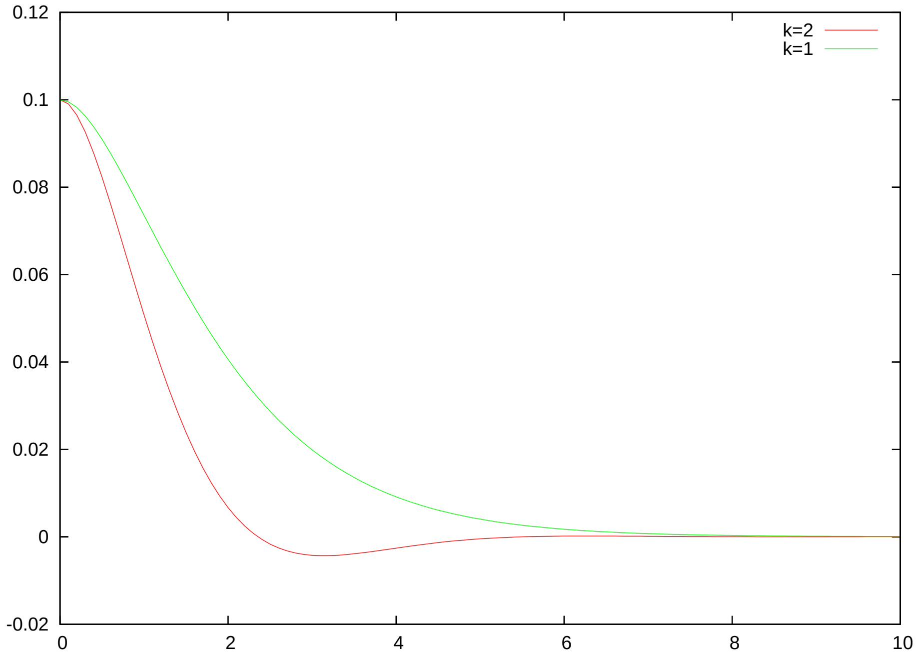
(c) The sketch should show a slightly oscillatory response, starting at $x(0) = 0.1$ with decreasing amplitude as t increases (red '$k = 2$' line in the plot below). Obtaining A and B from the initial conditions, $x(0) = 0.1$ and $\dot{x}(0) = 0$, the exact equation of the curve is

$$x(t) = 0.1e^{-t}\left(\sin(t) + \cos(t)\right)$$

but this is not necessary for full marks

(d) The reduction in spring constant will make the system critically damped, thus stopping it from oscillating (green '$k = 1$' line below). The exact equation (not required) is

$$x(t) = 0.1(1 + t)e^{-t}$$



0.12
0.1
0.08
0.06
0.04
0.02
0
-0.02
0
2
4
6
8
10
k=2
k=1